# Number T(n, k) of ways of arranging k non-attacking semi-queens on a n X n toroidal board.

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k and n are non-negative integers with $0 \le k \le n$.

$T(0, 0) := 1$    (for combinatorical reasons)

A semi-queen can only move horizontal, vertical and parallel to the main diagonal of the board. Moves parallel to the secondary diagonal are not allowed.

Instead of a board on a torus, you can imagine that the semi-queens can leave a flat board on one side and re-enter the board on the other side.

**Some properties of T(n, k)**

(1) $T(n, 0) = 1$

(2) $T(n, 1) = n^2$

(3) $T(n, 2) = n^2 \cdot (n-1)(n-2) / 2$

(4) $T(n, 3) = n^2 \cdot (n-1)(n-2) \cdot [(n-2)(n-4) + 2] / 6$

(5) $T(n, n) = 0$    if n is even and n > 0

$T(n, n) =$ A006717 $((n-1)/2)$    if n is odd

(6) $T(n, n-1) = n \cdot T(n, n)$    if n is odd

(7) $T(n, k) \equiv 0 \bmod n$    for $0 < k \le n$

**Proof sketches and remarks:**

(1) k = 0: The empty board is unique.

(2) k = 1: There are $n^2$ squares (cells) in which the queen can be placed.

(3) k = 2: For the second queen there are (n – 2) free cells in each row, except of the row with the first queen. Thus (n – 1) (n – 2) cells are free. We have to devide by 2 because the queens can be interchanged.

(4) Further considerations show that for the third queen there are $(n-2)(n-4) + 2$ free cells. Then division by $3! = 6$ is necessary because 3! permutations of three queens are possible.

--- What about T(n, 4)? The number of free cells for the fourth queen depends on the arrangement of the first three queens. Therefore an easy formula is not available.

(5) This equation can be proved similar like the proposition in the Lecture Latin Squares and the n-Queens Problem by Padriaic Bartlett.
You just have to use another circulant Latin Square namely:

| | | | |
|---|---|---|---|
| 0 | 1 | ⋯ | n – 1 |
| n – 1 | 0 | ⋯ | n – 2 |
| ⋮ | ⋮ | ⋱ | ⋮ |
| 1 | 2 | ⋯ | 0 |

This Latin Square does not have any transversals, if n is even.
It would be orthogonal to any semi-pandiagonal Latin Square of order n.
But as each Latin Square which has an orthogonal mate can be broken up into disjoint transversals there cannot exist a semi-pandiagonal Latin Square of even order. From any solution of the semi-queen problem with k = n we would be able to construct a semi-pandiagonal Latin Square. Thus such a solution does not exist for even n, except of the case n = 0.

(6) A solution for k = n – 1 can be found by taking away an arbitrary queen of a solution for k = n. This method leads to unique solutions for k = n – 1, because (n – 1) queens attack (n – 1) rows and (n – 1) columns on the board, they leave at most one cell free.

(7) From any solution you can derive another solution by cyclic permutation of the columns. Thus the number of non-empty (k > 0) solutions is a multiple of the number of columns n. In many cases the Number of solutions is a multiple of $n^2$:
k = 1, k = 2, k = 3, k = n – 1. See conjectures.

**Conjectures for positive n and k**

| | | | |
|---|---|---|---|
| (8) | $T(n, k) \equiv (k-2)n^2/k$ | $\bmod n^2$ | if k is an odd prime and $n \equiv 0 \bmod k$ |
| | $T(n, k) \equiv 0$ | $\bmod n^2$ | otherwise |
| (9) | $T(n, k) \equiv 0 \bmod 3$ | if $n \equiv 0$ or $k \equiv 2 \bmod 3$ | |
| (10) | $T(n, k) \equiv 1 \bmod 2$ | if $n \equiv -1 \bmod 2^a$ and $k > n - a$ | |

These conjectures are true for n < 20. See tables at the end of the paper.

**Algorithm**

First look at the possible distributions of k non-empty rows among the n rows of the board. Two distributions are equivalent if they can be transformed into each other by cyclic permutation of the rows or by reversing the order of the rows. From each equivalent class we have to consider only one representative distribution. We choose one where the first row is not empty. The maximum number of distributions of the same class is 2n.

For k > 2 we call the following recurrence procedure Row(y) with y = 2. We can assume that in the first row a queen is in the leftmost cell (factor of reduction: n).
Remp(y) = 1 means that this row has to be left empty, otherwise Remp(y) = 0.
stac(x) is the status of column x, where stac(x) = 1 means that the column is free.
stad(d) is the status of diagonal d with d = (x – y + n) mod n; thus d = 0, 1, 2, ..., n – 1.

```
Proc Row(y)
  If y > n
    cnt++
  Else If Remp(y)
    Row(y + 1)
  Else
    Local int x = 1
    While x < n : x++
      If stac(x)
        Local int d = x - y
        If d < 0 Then d += n
        If stad(d)
          stad(d) = 0 : stac(x) = 0
          Row(y + 1)
          stad(d) = 1 : stac(x) = 1
```

**Applications**

Constructing of certain kinds of Latin Squares and calculation of their amount.

This will be discussed in another paper by Hans-Bernhard Meyer and me.
A link to this paper will appear here.

## Tables

| T(n,k) | k = 0 | k = 1 | k = 2 | k = 3 | k = 4 | k = 5 | k = 6 | k = 7 | k = 8 | k = 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| n = 0 | 1 | | | | | | | | | |
| n = 1 | 1 | 1 | | | | | | | | |
| n = 2 | 1 | 4 | 0 | | | | | | | |
| n = 3 | 1 | 9 | 9 | 3 | | | | | | |
| n = 4 | 1 | 16 | 48 | 32 | 0 | | | | | |
| n = 5 | 1 | 25 | 150 | 250 | 75 | 15 | | | | |
| n = 6 | 1 | 36 | 360 | 1200 | 1224 | 288 | 0 | | | |
| n = 7 | 1 | 49 | 735 | 4165 | 8869 | 6321 | 931 | 133 | | |
| n = 8 | 1 | 64 | 1344 | 11648 | 43136 | 64512 | 33024 | 4096 | 0 | |
| n = 9 | 1 | 81 | 2268 | 27972 | 160866 | 423306 | 469800 | 188568 | 18225 | 2025 |
| n = 10 | 1 | 100 | 3600 | 60000 | 497200 | 2055360 | 4058400 | 3484800 | 1101600 | 92800 |
| n = 11 | 1 | 121 | 5445 | 117975 | 1334630 | 8024478 | 25116938 | 38699430 | 26511705 | 6774185 |
| n = 12 | 1 | 144 | 7920 | 216480 | 3211200 | 26574336 | 121936320 | 300024576 | 371458944 | 208106496 |
| n = 13 | 1 | 169 | 11154 | 375518 | 7078565 | 77341329 | 491658180 | 1784414892 | 3548710503 | 3610721335 |
| n = 14 | 1 | 196 | 15288 | 621712 | 14525560 | 202897632 | 1712877712 | 8661499840 | 25548854016 | 41984905984 |
| n = 15 | 1 | 225 | 20475 | 989625 | 28080675 | 488700135 | 5300844525 | 35768774475 | 147684079575 | 361572458175 |
| n = 16 | 1 | 256 | 26880 | 1523200 | 51613184 | 1095929856 | 14876990464 | 129544470528 | 716788205568 | 2466717499392 |
| n = 17 | 1 | 289 | 34680 | 2277320 | 90848884 | 2312944452 | 38458305080 | 420707308648 | 3015129321270 | 13955309953126 |
| n = 18 | 1 | 324 | 44064 | 3319488 | 154024416 | 4633324416 | 92698241616 | 1246387839552 | 11259962997984 | 67728606461568 |
| n = 19 | 1 | 361 | 55233 | 4731627 | 252698556 | 8870068908 | 210343699608 | 3413503220712 | 38017266879546 | 289100745522490 |
| n = 20 | 1 | 400 | 68400 | 6612000 | 402748800 | 16318909440 | 452845689600 | 8734923673600 | 117740009798400 | 1106795506739200 |
| n = 21 | 1 | 441 | 83790 | 9077250 | 625573935 | 28985523651 | 930868965180 | 21064414900788 | 338320205076546 | 3858438313852290 |
| n = 22 | 1 | 484 | 101640 | 12264560 | 949535400 | 49896376992 | 1836701481648 | 48211381444160 | 910475843856480 | 12400568405329280 |
| n = 23 | 1 | 529 | 122199 | 16333933 | 1411660305 | 83514794253 | 3493934459631 | 105341758435353 | 2312430325711770 | ? |
| n = 24 | 1 | 576 | 145728 | 21470592 | 2059643520 | 136290806784 | 6431958240768 | 220826072208384 | 5578572477413376 | ? |
| n = 25 | 1 | 625 | 172500 | 27887500 | 2954173750 | 217374984750 | 11494976237500 | 445975937012500 | 12851885391534375 | ? |

| T(n,k) | k = 10 | k = 11 | k = 12 | k = 13 | k = 14 | k = 15 |
|---|---|---|---|---|---|---|
| n = 10 | 0 | | | | | |
| n = 11 | 416361 | 37851 | | | | |
| n = 12 | 44024832 | 2525184 | 0 | | | |
| n = 13 | 1683797362 | 300019278 | 13394771 | 1030367 | | |
| n = 14 | 35768852224 | 14118547968 | 2145256064 | 88987136 | 0 | |
| n = 15 | 499636145925 | 361741341375 | 122420762325 | 16090554825 | 545443875 | 36362925 |
| n = 16 | 5099340857344 | 6013211836416 | 3748248092672 | 1100426510336 | 126087069696 | 4000841728 |
| n = 17 | 40692618294088 | 72028514448312 | 73361419855316 | 39787833739332 | 10227067821192 | 1032626877976 |
| n = 18 | 266489684190720 | 667515378760704 | 1023809354545536 | 910191600677376 | 433602198116352 | 98432337991680 |
| n = 19 | 1482534943682350 | 5024983150870218 | 10936355229760020 | 14680529993427396 | 11493936400932072 | 4848254835976920 |
| n = 20 | 7194807928704000 | 31867338417254400 | ? | ? | ? | ? |

| T(n,k) | k = 16 | k = 17 | k = 18 | k = 19 | k = 20 | k = 21 |
|---|---|---|---|---|---|---|
| n = 16 | 0 | | | | | |
| n = 17 | 27302144721 | 1606008513 | | | | |
| n = 18 | 8814325049856 | 220046616576 | 0 | | | |
| n = 19 | 979143776168997 | 78363862335261 | 1665481040929 | 87656896891 | | |
| n = 20 | ? | 10072974881587200 | 724233299558400 | 14625439744000 | 0 | |
| n = 21 | ? | ? | ? | 6952554432730998 | 121340556023715 | 5778121715415 |

| T(n,k) | k = n – 1 | k = n |
|---|---|---|
| n = 22 | 1152879813910528 | 0 |
| n = 23 | 10414280336082195 | 452794797220965 |
| n = 24 | ? | 0 |
| n = 25 | 104023922973515625 | 41609568918940625 |