Proofs of the claims in the Comments section of Ayyyyyy

Hartmut F. W. Hoft, Oct 30, 2020

Throughout only odd numbers n = $\prod p_i^{e_i}$ , $p_i$ distinct odd primes and $e_i$ positive, that have at least two prime factors are considered; the total number of divisors of n, 1 = $d_1 < d_2 < \dots < d_{\sigma_0(n)}$ = n, is $\sigma_0$(n) = $\prod(e_i + 1)$. The symmetric representation of $\sigma$(n), i.e. the list of the areas of its regions between two adjacent Dyck paths, is denoted by srs(n), see A237270 & A237271, and its total area by area(srs(n)). Functions with two arguments, such as a237048(n,k), denote the value of the k-th entry in the n-th row of the respective irregular triangle of the sequence referenced, i.e. A237048; when a single argument is used, such as a237048(n), it represents the list of values in the entire n-th row. All triangles referenced have the same shape; their n-th row has row(n) = $\lfloor \left(\sqrt{8n+1} - 1\right)/2 \rfloor$ many entries so that the 2-nd indices in functions are assumed to be in the range 1...row(n)+1.

t(n,k) = a235791(n,k) = $\lceil \frac{n+1}{k} - \frac{k+1}{2} \rceil$ and a235791(n,row(n)+1) = 0 in the triangle of A235791.

leg(n,k) = a237591(n,k) = t(n,k) - t(n,k+1) is the length of the k-th segment of the n-th Dyck path; legs(n) represents the entire n-th row.

a237048(n,k) = 1 when k|n or when k = 2 × s where s|n and 2 × s ≤ row(n) < $\frac{n}{s}$ , otherwise a237048(n,k) = 0 ; thus all (odd) divisors of n are represented by 1's in the n-th row of the triangle of A237048.

width(n,k) = a249223(n,k) = $\sum_{j=1}^{k} (-1)^{j+1}$ a237048($n$, $j$) is the width of the k-th leg between the n-th and (n-1)-st Dyck paths; widths(n) represents the entire n-th row of widths and diag(n) = width(n,row(n)) is the area of the squares between the Dyck paths containing the diagonal.

From A249223 we have: area(srs(n)) = 2× legs(n) . widths(n) - diag(n) (1)

where "." denotes the inner product.

The length of a symmetric central region starting with leg s is :

$2 \times \sum_{i=s}^{\text{row}(n)} \text{leg}(n, i) - 1 = 2 \times \sum_{i=s}^{\text{row}(n)} (t(n, i) - t(n, i+1)) - 1 = 2 \times t(n,s) - 1$ (2)

Since a237048(n,1) = a237048(n,2) = 1 it follows that width(n,1) = 1 and width(n,2) = 0, i.e. the first and last region of srs(n) each consists of a single leg of width 1 and length (hence area) leg(n,1) = $\lceil \frac{n+1}{2} \rceil$.

Therefore, srs(n) consists of 3 regions with a center region of maximum width 2 when widths(n) has the following numeric pattern in the triangle of A249223:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| position index in row: | 1 | 2 | | $d_2$ | $d_3$ | $2 \times d_2$ | $d_4$ | $2 \times d_3$ | $d_5$... |
| value at position: | 1 | 0 | ... 0 | 1 ... 1 | 2 ... 2 | 1 ... 1 | 2 ... 2 | 1 ... 1 | 2 ... (3) |
| divisor represented: | $d_1$ | $\frac{n}{1}$ | | $d_2$ | $d_3$ | $\frac{n}{d_2}$ | $d_4$ | $\frac{n}{d_3}$ | $d_5$ ... |

Lemma 1:
When srs(n) consists of 3 regions of maximum width 2 the center region contains $\sigma_0(n)$ - 3 areas of width 2. An area including the diagonal has width 2 when $\sigma_0(n)$ is even.

Lemma 2:
The central region of srs(n) = srs( $3^e \times 5$ ), e ≥1, has 2×e - 1 areas of width 2.

Lemma 3:
45 = 5 × $3^2$ is the only odd number in the second column of the first table.

Lemma 4:
For n = p × q, a product of distinct odd primes, srs(n) consists of 3 regions of maximum width 2 precisely when p < q < 2 × p.

Lemma 5:
The area of srs(p × q) = ( $\frac{p \times q+1}{2}$ , p + q , $\frac{p \times q+1}{2}$) , where p and q are primes and p < q < 2 × p, equals $\sigma$(p × q). Furthermore, the central region consists of two symmetric subparts of width 1 of length 2 × q - p and 2 × p - q.

Proof of Lemma 1:
When for some index h, $d_h \leq$ row(n) < 2 × $d_{h-1}$ then n has an even number of divisors, h = $\frac{\sigma_0(n)}{2}$ + 1, and by pattern (3) the center region of srs(n) has 2×(h - 3) + 1 = $\sigma_0(n) - 3$ areas of width 2. Similarly, when 2 × $d_{h-1} \leq$ row(n) < $d_{h+1}$ then n has an odd number of divisors, h = $\frac{\sigma_0(n)-1}{2}$ + 1, and by pattern (3) the center region of srs(n) has 2×(h - 2) = $\sigma_0(n)$ - 3 areas of width 2. An alternative description of pattern (3) is: 1 = $d_1 < d_2 < d_3$ and $d_i < 2 \times d_{i-1} < d_{i+1}$ , for 3 ≤ i ≤ $\lfloor \frac{\sigma_0(n)}{2} \rfloor$ + 1 = h.

□

Proof of Lemma 2:
1 < 2 < 3 < 5 < 2×3 < $3^2$ < 2×5 < b(1) < ... < b(i) < 2×$3^{i+1}$ < $3^{i+2}$ < 2×b(i) < b(i+1) < ... , for i ≥ 1, shows that pattern (3) for widths is satisfied so that $\sigma_0(3^e \times 5)$ - 3 = 2×(e + 1) - 3 = 2×e - 1, for e ≥ 1.
In addition, pattern (3) requires for an even number $2^k$× q, k ≥ 1, q odd, that its two smallest proper divisors must satisfy 4 ≤ $2^{k+1} < d_2 < d_3 < 2^{k+1} \times d_2$, and for an odd number that its two smallest proper divisors must satisfy 2 < $d_2 < d_3 < 2 \times d_2$ so that the numbers b(i) are the smallest numbers in their respective columns.

□

Proof of Lemma 3:
We eliminate the three possible patterns for prime factors of n:
p × $q^2$, except for p = 3 & q = 5; $p^3$ ; p × q × r , distinct primes.

(a) Let n = p × $q^2$, with p < $q^2$ and p, q distinct odd primes. Then pattern (3) is fulfilled precisely when
q < p < 2 × q < $q^2 \leq \left\lfloor \left( \sqrt{8 \times p \times q^2 + 1} - 1 \right) / 2 \right\rfloor$ < 2 × p (*)
since p < q < 2 × p < p × q < 2 × q - satisfying pattern (3) - is a contradiction.
Therefore, (*) implies $q^2$ < 2 × p < 4 × q , i.e. q = 3 and p = 5.
$q^2$ < p would imply q < $q^2$ < 2 × q < p to satisfy pattern (3), contradicting that q is prime.

(b) Let $n = p^3$, with p an odd prime. Then $1 < 2 < p < 2 \times p < p^2$ implies that srs($p^3$) consists of 4 regions (see also A280107).

(c) Let $n = p \times q \times r$ with $p < q < r$ distinct odd primes. If $r < p \times q$ then pattern (3) provides $p < q < 2 \times p < r < 2 \times q < p \times q \leq \text{row}(p \times q \times r) < 2 \times r$,
but $r < 2 \times q$ together with $p \geq 5$ implies $2 \times r < 4 \times q < p \times q < 2 \times r$, a contradiction. Therefore, p = 3 which forces q = 5 and $r < 2 \times q = 10$, i.e. r = 7, so that both $10 = 2 \times q$ and $14 = 2 \times r$ are smaller than 15 $= p \times q$ implying that srs(3×5×7) has 4 regions.
For the inequality $p < q < p \times q < r$ pattern (3) leads to the contradiction
$p < q < 2 \times p < p \times q < 2 \times q$.

□

Proof of Lemma 4:
Pattern (3) is fulfilled precisely when
$q \leq \left\lfloor \left(\sqrt{8 \times p \times q + 1} - 1\right)/2 \right\rfloor < 2 \times p$
$\Leftrightarrow \quad 2 \times q + 1 \leq \sqrt{8 \times p \times q + 1} < 4 \times p + 1$
$\Leftrightarrow \quad 4 \times q^2 + 4 \times q + 1 \leq 8 \times p \times q + 1 < 16 \times p^2 + 8 \times p + 1$
$\Leftrightarrow \quad q + 1 \leq 2 \times p$

□

Proof of Lemma 5:
Pattern (3) for $n = p \times q$ is:

| | | | | | |
|---|---|---|---|---|---|
| position in row: | 1 | 2 | p | $q$ | row(p×q) < 2×$p$ |
| value at position: | 1 | 0 ... 0 | 1 ... 1 | 2 ... | 2 ... |
| divisor represented: | 1 | p×q | p | q | |

Therefore, $\text{width}(n,k) = \begin{cases} 1 & \text{for } k = 1, p \ldots q-1 \\ 0 & \text{for } k = 2 \ldots p-1 \\ 2 & \text{for } k = q \ldots \text{row}(n) \end{cases}$

$\text{leg}(n,1) = \frac{p \times q + 1}{2}$ and with its width of 1 establishes the area of each outer region.
By formula (2) the length of the entire symmetric central region is
$2 \times t(n,p) - 1 = 2 \times \left\lceil \frac{p \times q + 1}{p} - \frac{p+1}{2} \right\rceil - 1 = (2 \times q - (p + 1) + 2) - 1 = 2 \times q - p.$
Similarly, the length of the symmetric extent of width 2 in the central region is
$2 \times t(n,q) - 1 = 2 \times \left\lceil \frac{p \times q + 1}{q} - \frac{q+1}{2} \right\rceil - 1 = (2 \times p - (q + 1) + 2) - 1 = 2 \times p - q.$
Therefore, $\text{area}(\text{srs}(n)) = 2 \times \frac{p \times q + 1}{2} + (2 \times q - p) + (2 \times p - q) = p \times q + p + q + 1 = \sigma(n)$.

□