RANDOM WALK ON THE SQUARE LATTICE: RETURN TO (0,0) WITH OR WITHOUT PASSING (1,0)

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Abstract. In the simple random walk on the 2D square lattice, the probability of returning to the origin is 1. We derive the probability of 1/2 of first returning to the origin without passing through the vertex (1,0) and the probability of 1/2 of first passing through the vertex (1,0) before returning to the origin.

1. Notations

We consider simple random walks on the infinite square lattice that start at the origin and take n steps with equal probability $\frac{1}{4}$ to any of the 4 adjacent vertices. *Simple* means there are no constraints of staying in some quadrants of the lattice, no constraints on selfavoidance and no constraints of walking back to the vertex of the previous step. Chains of the following mnemonics serve to trace a walk by the type of vertex after some number of steps:

- 0 is a step on the origin;
- 1 is a step on the $(1, 0)$ vertex right from the origin;
- X is a step on a vertex that is neither a 0 nor a 1;
- $\bar{0}$ is a step not on the origin;
- $\bar{1}$ is a step not on 1 (the union of 0 and X);
- $*$ is a step on any vertex (the union of 0, 1 and X).

Upper indices indicate paths which remain in one set of vertices (similar to frequency indicators of partition notations). We add a hash in front of a chain of mnemonics for the count of random walks of that category.

Because each step changes either the x or the y-coordinate of the vertex by 1, a parity rule applies: the number of walks of length n starting at the origin to some vertex is zero if the parity (remainder after division through 2) of n does not equal the remainder of the sum of the x and y coordinate of the vertex.

A walk can also be described by a word of length n of the 4-letter alphabet L(eft), R(ight), U(p) and D(own) describing which compass direction of the square lattice a step takes.

The basic examples of enumerating walks are:

- There is one walk of length 1 ending at 1. $\#(01) = 1$.
- There are four walks of length 2 returning to 0: UD, LR, RL, DU, $\#(0\bar{0}0) = 4$, see the leading term in (3).
- There is are two walks of length 3 avoiding 0 and 1 at intermediate steps and ending at 1: URD and DRU, $\#(0X^2 1) = 2$.

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- There is are five walks of length 3 avoiding 0 and ending at 1: URD, DRU, RRL, RUD, and RDU, $\#(0\bar{0}^2 1) = 5$.
- There is are nine walks of length 3 ending at 1: URD, DRU, RRL, RUD, RDU, RLR, LRR, UDR, DUR $\#(0{*}1) = 9$. This is a preview of a coefficient in Eq. (6).
- There are 16 walks of length 5 avoiding 0 and 1 at intermediate steps and ending at 1: DDRUU, DRDUU, DDURU, LDRRU, DLRRU, DRLRU, DR-RLU, LURRD, ULRRD, UUDRD, URLRD, URUDD, UURDD, URRLD, DRRUL, and URRDL: $\#(0X^4 1) = 16$. This is a preview of a coefficient of Eq. (18).

2. Returns

A walk of length n that that returns to the origin is a word of length n on the 4-letter alphabet where the number of (letters of) L equals the number of R, and where the number of U equals the number of D. By the parity rule the number of these walks is zero if n is odd. For n even we may distribute the L without constraint over the n places, then distribute the R over the remaining $n-L$ places, then distribute the $U = (n-2L)/2$ over the remaining $n-2L$ places. So the number of walks that return to the origin is

(1)
$$\#(0{*}^{n-1}0) = \sum_{L=0}^{n/2} \binom{n}{L}\binom{n-L}{L}\binom{n-2L}{n/2-L} = \sum_{L=0}^{n/2} \frac{n!}{(n-L)!L!}\frac{(n-L)!}{(n-2L)!L!}\frac{(n-2L)!}{[(n/2-L)!]^2}$$
$$= n!\sum_{L=0}^{n/2} \frac{1}{[L!]^2}\frac{1}{[(n/2-L)!]^2} = \frac{n!}{[(n/2)!]^2}\sum_{L=0}^{n/2}\left[\frac{(n/2)!}{L!(n/2-L)!}\right]^2$$
$$= \frac{n!}{[(n/2)!]^2}\sum_{L=0}^{n/2}\left[\binom{n/2}{L}\right]^2 = \left[\binom{n}{n/2}\right]^2.$$

This is sequence [4, A002894] in the Online Encyclopedia of Integer Sequences. The summation is a special case for sums of squares of binomial coefficients [9, 8].

Define the generating function

$$\sum_{n=0,2,4,\ldots} \#(0 *^{n-1} 0)z^n \equiv \#_z(0 *^{n-1} 0) = \sum_{l=0,1,2,\ldots}\left[\frac{(2l)!}{l!^2}\right]^2 z^{2l} \tag{2}$$
$$= {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;(4z)^2\right) = 1 + 4z^2 + 36z^4 + 400z^6 + \ldots.$$

for these returns by a standard technique to sum up hypergeometric series [6]. The number of first returns after n steps, $\#(0\bar{0}^{n-1}0)$, has a generating function which is the series inversion [1, 3]

$$\sum_{n=0,2,4,\ldots} \#(0\bar{0}^{n-1}0)z^n \equiv \#_z(0\bar{0}^{n-1}0) = 1 - \frac{1}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;(4z)^2\right)} \tag{3}$$
$$= 4z^2 + 20z^4 + 176z^6 + 1876z^8 + \cdots.$$

This is sequence [4, A054474]. The key point of the following analysis is that we consider these numbers $\#(0\bar{0}^{n-1}0)$, number of walks with first return to the origin after n steps, to be perfectly known through this generating function tracer.

Dividing the number of walks of length n through 4^n assigns a probability to each walk. In particular the probability of a walk of length n to return for the first time to the origin is $\#(0\bar{0}^{n-1}0)/4^n$. By summation over n, the total probability of all walks to return for the first time to 0 is the generating function at $z = 1/4$, where

$$1 - \frac{1}{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; 1)} = 1. \tag{4}$$

The return is certain for this type of walks on the 2D square lattice [5].

3. Passage through 1

A walk of length n that ends at 1 is a word of length n where the number of R is one larger than the number of L, and where the number of U equals the number of D: $R = 1 + L$, $R + L + U + D = n$, $U = D$. By the parity criterion this is zero if n is even. If n is odd we repeat the combinatorics of first selecting the places of the L's, then the R's and then the U's as above:

$$\begin{aligned}
\#(0 *^{n-1} 1) &= \sum_{L=0}^{(n-1)/2} \binom{n}{L}\binom{n-L}{L+1}\binom{n-2L-1}{(n-1)/2-L} \qquad (5)\\
&= \sum_{L=0}^{(n-1)/2} \frac{n!}{(n-L)!L!}\,\frac{(n-L)!}{(n-2L-1)!(1+L)!}\,\frac{(n-2L-1)!}{[(n-1)/2-L]!((n-1)/2-L)!}\\
&= \binom{n}{(n-1)/2}\sum_{L=0}^{(n-1)/2}\binom{(n-1)/2}{L}\binom{(n+1)/2}{L} = \frac{1}{2}\binom{n}{(n-1)/2}\frac{(n+1)!}{[(n+1)/2]^2}\\
&= \left[\binom{n}{(n-1)/2}\right]^2 .
\end{aligned}$$

This is sequence [4, A060150]. The associated generating function is

$$\begin{aligned}
\sum_{n=1,3,5,\ldots} \#(0 *^{n-1} 1) z^n \equiv \#_z(0 *^{n-1} 1) &= z\,{}_3F_2\left(1, \frac{3}{2}, \frac{3}{2}; 2, 2; (4z)^2\right) \qquad (6)\\
&= z + 9z^3 + 100z^5 + 1225z^7 + 15876z^9 \cdots .
\end{aligned}$$

4. Return to origin avoiding 1

The number of walks that return to the origin avoiding 0 during all intermediate steps is $\#(0\bar{0}^{n-1}0)$, and can be reclassified by the walks that pass by 1 never, once, twice, . . . times in between [10]:

(7)

$$\#(0\bar{0}^{n-1}0) = \#(0X^{n-1}0) + \sum_i \#(0X^i1X^{n-i-2}0) + \sum_{i,j} \#(0X^i1X^j1X^{n-i-j-3}0) + \cdots$$

Because the sections of the walks arriving and leaving at 1 are independent (i.e.,the walks have no memory), these numbers are multiplicative:

$$\#(0\bar{0}^{n-1}0) = \#(0X^{n-1}0) + \sum_i \#(0X^i1)\#(1X^{n-i-2}0) + \sum_{i,j} \#(0X^i1)\#(1X^j1)\#(1X^{n-i-j-3}0) + \cdots \tag{8}$$

Furthermore

- the number of walks is independent from the direction, which means leaving 1 and arriving at 0 counts also leaving 0 and arriving at 1 (if the avoidances are equivalent/swapped). These walks are bijections if each walk is mirrored along the vertical line $x = 1/2$.
- the number of returns is shift-independent from the origin, which means leaving 1 and arriving at 1 is the same as leaving 0 and arriving at 0 (if avoidances are swapped accordingly).

Therefore

$$\#(0\bar{0}^{n-1}0) = \#(0X^{n-1}0) + \sum_i \#(0X^i1)\#(0X^{n-i-2}1) + \sum_{i,j} \#(0X^i1)\#(0X^j0)\#(0X^{n-i-j-3}1) + \cdots \tag{9}$$

Multiplication by z^n and summation over n induces for the generating functions a geometric series:

$$\#_z(0\bar{0}^{n-1}0) = \#_z(0X^{n-1}0) + [\#_z(0X^{n-1}1)]^2 + [\#_z(0X^{n-1}1)]^2\#_z(0X^{n-1}0) + [\#(0X^{n-1}1)]^2[\#_z(0X^{n-1}0)]^2 + \cdots = \#_z(0X^{n-1}0) + \frac{[\#_z(0X^{n-1}1)]^2}{1 - \#_z(0X^{n-1}0)} \tag{10}$$

5. Pasage to 1 avoiding 0 and 1

We classify the walks from 0 to 1 as walks that either first step on 1 or first step on 0, and reorganize the count of trails with the criterion of mirror-symmetry and shift-invariance as above:

$$\begin{aligned}
\#(0*^{n-1}1) &= \#(0X^i1*^{n-2-i}1) + \#(0X^i0*^{n-2-i}1) \\
&= \sum_i \#(0X^i1)\#(1*^{n-2-i}1) + \sum_i \#(0X^i0)\#(0*^{n-2-i}1) \\
&= \sum_i \#(0X^i1)\#(0*^{n-2-i}0) + \sum_i \#(0X^i0)\#(0*^{n-2-i}1). \qquad (11)
\end{aligned}$$

Moving on to the generating functions replaces convolution sums by simple products:

$$\#_z(0*^{n-1}1) = \#_z(0X^{n-1}1)\#_z(0*^{n-1}0) + \#_z(0X^{n-1}0)\#_z(0*^{n-1}1). \tag{12}$$

$$\therefore [1 - \#_z(0X^{n-1}0)]\#_z(0*^{n-1}1) = \#_z(0X^{n-1}1)\#_z(0*^{n-1}0). \tag{13}$$

Isolate $1-\#_z(0X^{n-1}0)$ of the left hand side of this equation and insert it into the denominator in (10):

$$\#_z(0\bar{0}^{n-1}0) = \#_z(0X^{n-1}0) + \#_z(0X^{n-1}1)\frac{\#_z(0*^{n-1}1)}{\#_z(0*^{n-1}0)}. \tag{14}$$

Regard the previous 2 equations as a 2×2 system of linear equations for two unknown generating functions:

$$\#_z(0X^{n-1}1) \quad = \quad = \frac{1}{2(a_0-a_1)} - \frac{1}{2(a_0+a_1)}; \tag{15}$$

$$\#_z(0X^{n-1}0) \quad = \quad = 1 - \frac{1}{2(a_0+a_1)} - \frac{1}{2(a_0-a_1)}, \tag{16}$$

where $a_0 = \#_z(0*^{n-1}0)$ is given by (2) and $a_1 \equiv \#_z(0*^{n-1}1)$ is given by (6). This result was already obtained by Rubin and Weiss [7, (32)]. Explicit insertion of the two hypergeometric series gives sequences [4, A337869,A337870]:

$$\#_z(0X^{n-1}0) = 3z^2+13z^4+106z^6+1073z^8+12142z^{10}+147090z^{12}+1865772z^{14} + 24463905z^{16}+328887346z^{18}+\cdots, \tag{17}$$

$$\#_z(0X^{n-1}1) = z+2z^3+16z^5+166z^7+1934z^9+24076z^{11}+312906z^{13} + 4191822z^{15}+57433950z^{17}+800740450z^{19}+\cdots. \tag{18}$$

Due to poles of the hypergeometric functions both $a_0\to\infty$ and $a_1\to\infty$ as $z\to 1/4$, so for the probability of either passing through 0 or through 1 we get the complementary

$$\#_z(0X^{n-1}0) \quad \to \quad 1-\frac{1}{2(a_0-a_1)}; \tag{19}$$

$$\#_z(0X^{n-1}1) \quad \to \quad \frac{1}{2(a_0-a_1)}. \tag{20}$$

In a final manœvre we calculate that both probabilities are $1/2$ as $z\to 1/4$, because in the denominators

$$\begin{aligned} a_0-a_1 &= \sum_{n=0,2,4,\ldots}[\binom{n}{n/2}]^2z^n - \sum_{n=1,3,5,\ldots}[\binom{n}{(n-1)/2}]^2z^n \\ &= 1+\sum_{n=1,2,3,\ldots}[\binom{2n}{n}]^2z^{2n} - \sum_{n=0,1,2,\ldots}[\binom{2n+1}{n}]^2z^{2n+1} \\ &= 1+\sum_{n\ge0}[\binom{2n+2}{n+1}]^2z^{2n+2} - \sum_{n\ge0}[\binom{2n+1}{n}]^2z^{2n+1} \\ &= 1+\sum_{n\ge0}[\binom{2n+2}{n+1}]^2/4^{2n+2} - \sum_{n\ge0}[\binom{2n+1}{n}]^2/4^{2n+1} = 1+\sum_{n\ge0}\frac{1}{4^{2n+2}}[\binom{2n+2}{n+1}]^2 - 4[\binom{2n+1}{n}]^2 \\ &= 1+\sum_{n\ge0}\frac{1}{4^{2n+2}}\left[\binom{2n+2}{n+1}+2\binom{2n+1}{n}\right]\underbrace{\left[\binom{2n+2}{n+1}-2\binom{2n+1}{n}\right]}_{=0} = 1. \end{aligned} \tag{21}$$

In summary

$$\#_{z=1/4}(0X^{n-1}0) = \#_{z=1/4}(0X^{n-1}1) = \frac{1}{2}. \tag{22}$$

Appendix A. Passage through $(k, 0)$

The walks of length n that end at the coordinate $(k, 0)$ are counted by words of length n with L left steps, $R = L + k$, $U = D$ and $R + L + U + D = n$. The cases $k = 0, 1$ have been evaluated in (2) and (6); the general counting numbers are [6]

(23)
$$\sum_{L=0}^{(n-k)/2} \binom{n}{L}\binom{n-L}{L+k}\binom{n-2L-k}{(n-k)/2-L} = \binom{n}{(n-k)/2} \sum_{L=0}^{(n-k)/2} \binom{(n-k)/2}{L}\binom{(n+k)/2}{L+k}$$
$$= \left[\binom{n}{(n-k)/2}\right]^2$$

if $n - k$ is even, otherwise 0. The generating functions are

$$a_k = \sum_{n=k,k+2,k+4,\ldots} [\binom{n}{(n-k)/2}]^2 z^n \tag{24}$$
$$= z^k {}_4F_3\left(\frac{k+1}{2}, \frac{k+1}{2}, 1+\frac{k}{2}, 1+\frac{k}{2}; 1, 1+k, 1+k; (4z)^2\right).$$

The analysis in the main text remains valid if the vertex $(k, 0)$ replaces the vertex $(1, 0)$ as the second marked vertex besides the origin, and a_k replaces a_1 in Eqs. (15) and (16).

References

1. Mira Bernstein and Neil J. A. Sloane, *Some canonical sequences of integers*, Lin. Alg. Applic. **226–228** (1995), 57–72, (E:) [2]. MR 1344554
2. Richard A. Brualdi, *From the editor-in-chief*, Lin. Alg. Applic. **320** (2000), no. 1–3, 209–216. MR 1796542
3. Philippe Flajolet and Robert Sedgewick, *Analytic combinatorics*, Cambridge University Press, 2009. MR 2483235
4. O. E. I. S. Foundation Inc., *The On-Line Encyclopedia Of Integer Sequences*, (2020), https://oeis.org/. MR 3822822
5. Barry D. Hughes, *On returns to the starting site in lattice random walks*, Physica **134A** (1986), no. 2, 443–457. MR 0828165
6. Ranjan Roy, *Binomial identities and hypergeometric series*, Amer. Math. Monthly **94** (1987), no. 1, 36–46. MR 0873603
7. Robert J. Rubin and George H. Weiss, *Random walks on lattices. the problem of visits to a set of points revisited*, J. Math. Phys. **23** (1982), 250–253. MR 0644245
8. Antonín Slavik, *Identities with squares of binomial coefficients*, Ars. Combin **113** (2014), 377–383. MR 3186480
9. H. A. Verrill, *Sums of squares of binomial coefficients, with applications to picard-fuchs equations*, arXiv:math/0407327 (2018).
10. George H. Weiss and Robert J. Rubin, *Random walks: Theory and selected applications*, Adv. Chem. Phys. **LII** (1983), 363–506.

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