# Lower Bounds for the Number of Primes in Some Integer Intervals

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**Abstract**: We showed that there is at least one prime number in the ranges of $(p, p+\pi(p)]$, $(p-\pi(p), p)$, $(n, n+\pi(n)]$ and $(n-\pi(n), n)$, and there are at least three prime numbers in the range of $(p-\pi(p),\ p+\pi(p)]$.

***<u>Theorem 1</u>: There is at least one prime number in the range of $(\boldsymbol{p}, \boldsymbol{p}+\boldsymbol{\pi}(\boldsymbol{p})]$, where $\boldsymbol{p}$ is a prime number and $\boldsymbol{\pi}(\boldsymbol{p})$ is the number of primes less than or equal to $\boldsymbol{p}$.***

***Proof:*** Let $N_p$ be the number of prime numbers in the range of $(p, p+\pi(p)]$, or

$$N_p := \pi\big(p+\pi(p)\big) - \pi(p) \tag{1}$$

To prove Theorem 1, we need to show $N_p \geq 1$.

Dusart [1] showed that the number of prime numbers less than or equal to x is bounded by

$$\pi(x) \geq \frac{x}{log\ x}\left(1+\frac{1}{\log x}+\frac{2}{\log^2 x}\right) \qquad \text{for } x > 88783 \tag{2a}$$

and

$$\pi(x) \leq \frac{x}{log\ x}\left(1+\frac{1}{\log x}+\frac{2.334}{\log^2 x}\right) \qquad \text{for } x > \ 2953652287. \tag{2b}$$

For $p > 2953652287$, a lower bound of $N_p$ can be determined based on Eqs. 2a and 2b.

$$N_p \geq \pi(p+\pi(p)_{min}) - \pi(p)_{max} \tag{3}$$

where

$$\pi(p)_{min} = \frac{p}{\log p}\left(1+\frac{1}{\log p}+\frac{a}{\log^2 p}\right) \tag{4a}$$

$$\pi(p)_{max} = \frac{p}{\log p}\left(1+\frac{1}{\log p}+\frac{b}{\log^2 p}\right) \tag{4b}$$

where a=2, b=2.334.

Let $q = p + \pi(p)_{min}$. From Eq. 3, we have

$$N_p \geq \pi(q) - \pi(p)_{max} \geq \pi(q)_{min} - \pi(p)_{max} \tag{5}$$

where

$$\pi(q)_{min} = \frac{q}{\log q}\left(1 + \frac{1}{\log q} + \frac{a}{\log^2 q}\right) \tag{6}$$

Substituting Eqs. 6 and 4b into Eq. 5 gives

$$N_p \geq \; q\left(\frac{1}{\log q} + \frac{1}{\log^2 q} + \frac{a}{\log^3 q}\right) - p\left(\frac{1}{\log p} + \frac{1}{\log^2 p} + \frac{b}{\log^3 p}\right). \tag{7}$$

Since

$$q = p + \pi(p)_{min} = p\left(1 + \frac{1}{\log p} + \frac{1}{\log^2 p} + \frac{a}{\log^3 p}\right), \tag{8}$$

from Eqs. 7 and 8, we have,

$$\frac{\mathrm{N_p}}{p} \geq \left[1 + \left(\frac{1}{\log p} + \frac{1}{\log^2 p} + \frac{\mathrm{a}}{\log^3 p}\right)\right]\frac{\log^2 q + \log q + a}{\log^3 q} - \left(\frac{1}{\log p} + \frac{1}{\log^2 p} + \frac{b}{\log^3 p}\right) \tag{9}$$

in which

$$\log q = \log p + \log\left(1 + \frac{1}{\log p} + \frac{1}{\log^2 p} + \frac{a}{\log^3 p}\right) < \log p + \frac{1}{\log p} + \frac{1}{\log^2 p} + \frac{a}{\log^3 p} \tag{10}$$

and thus

$$\frac{\log^2 q + \log q + a}{\log^3 q} > \frac{\log^2 p + \log p + (\mathrm{a}+2) + \frac{3}{\log p} + \frac{2(a+1)}{\log^2 p} + \frac{a+2}{\log^3 p} + \frac{2a+1}{\log^4 p} + \frac{2a}{\log^5 p} + \frac{a^2}{\log^6 p}}{\left(\log p + \frac{1}{\log p} + \frac{1}{\log^2 p} + \frac{a}{\log^3 p}\right)^3} > \frac{\log^2 p + \log p + (\mathrm{a}+2)}{\log^3 p} \tag{11}$$

Substituting Eq. 11 into Eq. 9 we get

$$\frac{N_p}{p} > \left[1 + \left(\frac{1}{\log p} + \frac{1}{\log^2 p} + \frac{a}{\log^3 p}\right)\right]\frac{\log^2 p + \log p + (\mathrm{a}+2)}{\log^3 p} - \left(\frac{1}{\log p} + \frac{1}{\log^2 p} + \frac{b}{\log^3 p}\right)$$

or

$$\frac{N_p}{p} > \frac{\log^4 p + (\mathrm{a}-\mathrm{b}+4)\log^3 p + 3\log^2 p + 2\log p - \mathrm{a}(\mathrm{a}+2)}{\log^6 p} > \frac{1}{\log^2 p} \tag{12}$$

Thus, for $p > 2953652287$, $N_p$, the number of prime numbers in the range of $(p, p + \pi(p)]$, has a lower bound of $\frac{p}{\log^2 p}$, which is greater than 1.

$$N_p > \frac{p}{\log^2 p} > 1 \tag{13}$$

In conclusion, for $> 2953652287$, there is at least one prime number in the range of $(p, p + \pi(p)]$. It can be verified that, for $2 \leq p \leq 2953652287, N_p \geq 1$. Therefore, Theorem 1, $\pi\big(p + \pi(p)\big) - \pi(p) \geq 1$, is proved. □

***Corollary 1: There is at least one prime number in the range of $(\boldsymbol{n}, \boldsymbol{n} + \boldsymbol{\pi}(\boldsymbol{n})]$, where n is an integer greater than or equal to 2 and $\boldsymbol{\pi}(\boldsymbol{n})$ is the number of primes less than or equal to n.***

***Proof:*** Let $n$ be an integer such that

$$p_m \leq n < p_{m+1} \tag{14}$$

where $p_m$ is the m-th prime number with $m \geq 1$ and $p_{m+1}$ the next prime number flowing $p_m$. By the definition of n,

$$\pi(n) = m \tag{15}$$

where $n \geq 2$. Theorem 1 tells us there exists at least one prime number in the range of $(p_m, p_m + m]$ and, by definition, there is exactly one prime number in the range $(p_m, p_{m+1}]$. Thus, $p_{m+1} \leq p_m + m$. Combining with Eq. 14, we have

$$p_m \leq n < p_{m+1} \leq p_m + m \tag{16}$$

Since $n \geq p_m$, adding $n$ on both sides of Eq. 15 gives

$$n + \pi(n) = n + m \geq p_m + m \tag{17}$$

As there is at least one prime number in the range of $(p_m, p_m + m]$ and $p_m + m \leq n + \pi(n)$, there must be at least one prime number in the range of $(p_m, n + \pi(n)]$.

By the definition of $n$, there is no prime number in the range of $(p_m, n]$. It can be concluded that there is at least one prime number in the range of $(n, n + \pi(n)]$, where $n \geq 2$. □

***Corollary 2: There is at least one prime number between $\mathbf{p} - \boldsymbol{\pi}(\mathbf{p})$ and $\mathbf{p}$, where $\mathbf{p}$ is a prime number greater than or equal to 3 and $\boldsymbol{\pi}(\mathbf{p})$ is the number of primes less than or equal to $\mathbf{p}$.***

***Proof:*** Let $k$ be the number of prime numbers less than or equal to $p_m - \mathrm{m}$, or

$$k = \pi(p_m - \mathrm{m}). \tag{18}$$

where m ≥ 2. Since

$$p_k \leq p_m - \mathrm{m} < p_m \tag{19}$$

which means $p_k < p_m$ and, thus, k < m. So, we have

$$p_k + k \leq p_m - m + k < p_m \tag{20}$$

According to Theorem 1, there is at least one prime number in the range of $(p_k, p_k + k]$ and, by the definition of $k$, there is no prime number in the range of $(p_k, p_m - m]$. Thus, there must be at least one prime number in the range of $(p_m - m, p_k + k]$.

Since $p_k + k < p_m$, there must be at least one prime number in the range of $(p_m - m, p_m)$. Let $p = p_m$ with m ≥ 2. We have $m = \pi(p)$. Therefore, it can be concluded that there is at least one prime number between $p - \pi(p)$ and $p$, where p ≥ 3. □

***Corollary 3: There is at least one prime number between $\mathbf{n} - \boldsymbol{\pi}(\mathbf{n})$ and $\mathbf{n}$, where n is an integer greater than or equal to 3 and $\boldsymbol{\pi}(\mathbf{n})$ is the number of primes less than or equal to n.***

***Proof:*** Let $n$ be an integer such that

$$p_{m-1} < n < p_m \tag{21}$$

in which $p_m$ is the m-th prime number with m > 2 and $p_{m-1}$ the previous prime number. By the definition of n,

$$\pi(n) = m - 1 \tag{22}$$

where n ≥ 3. Since

$$n - \pi(n) = n - (m - 1) = (n + 1) - m < (p_m + 1) - m \leq p_m - m \tag{23}$$

According to Corollary 2, there is at least one prime number in the range of $(p_m - m, p_m)$. So, there must be at least one prime number in the range of $(n - \pi(n), p_m)$.

By the definition of $n$, there is no prime number in the range of $[n, p_m)$. It can be concluded that there is at least one prime number between $n - \pi(n)$ and $n$, where n ≥ 3. □

***Corollary 4: There are at least three prime numbers in the range of $((\mathbf{p} - \boldsymbol{\pi}(\mathbf{p}), \mathbf{p} + \boldsymbol{\pi}(\mathbf{p})]$, where $\mathbf{p}$ is a prime number greater than or equal to 3 and $\boldsymbol{\pi}(\mathbf{p})$ is the number of primes less than or equal to $\mathbf{p}$.***

***Proof:*** From Corollary 2 and Theorem 1, we know that there is at least one prime number in the range of $(\mathrm{p} - \pi(\mathrm{p}), \mathrm{p})$ and $(\mathrm{p}, \mathrm{p} + \pi(\mathrm{p})]$. Since p is a prime number, there must be at least three prime numbers in the range of $(\mathrm{p} - \pi(\mathrm{p}), \mathrm{p} + \pi(\mathrm{p})]$. □

**References**

[1] P. Dusart, Estimates of some functions over primes without RH, arXiv:1002.0442v1 [math.NT] 2 Feb 2010.