Generalized harmonic series

$T(m,n)$ is the least k such that the partial sum of the series $H_m(k) = \sum_{h=0}^{k} \frac{1}{mh+1}$ is > n.

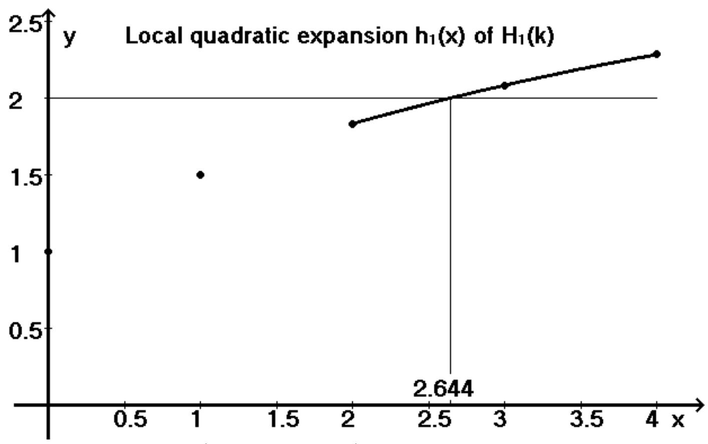
I) Formula for $T(m,n)$

I,1) Introduction

The differences $H_m(k) - n$, for $k = T(m,n)$, appear random, and it is very unlikely that a formula exists which describes these differences. However, if $H_m(k)$ is expanded continuously, we can solve the equation $H_m(k) = n$ with $k = k_0$ such that $T(m,n) = \lfloor k_0 + 1 \rfloor$. $T(m,1) = 1$ is evident and will not be considered any more.

I,2) Solution of a local quadratic expansion

One can use a local quadratic expansion $h_m(x)$, i.e. a quadratic function with $k = T(m,n)$, $h_m(k) = H_m(k)$ and $h_m(k \pm 1) = H_m(k \pm 1)$.
Let k_0 be the solution of $h_m(x) = n$.
$h_1(k) = 2$ is solved by $k_0 = 2.644$, see fig.



Generally:

(I,2) $k_0 = k + \frac{b}{2a} - \sqrt{\left(\frac{b}{2a}\right)^2 + \frac{H_m(k)-n}{a}}$

with $2a = \frac{1}{mk+1} - \frac{1}{m(k+1)+1}$ and $2b = \frac{1}{mk+1} + \frac{1}{m(k+1)+1}$

I,3) Solution of an asymptotic expansion (for large k_0, see IV)

(I,3a) $k_0 = k_1 - \frac{t}{24k_1}$ with $0 < t < 1$ and $k_1 = e^{mn-c(m)} - \frac{m+2}{2m}$

(I,3b) $t \to 1$ for $k_1 \to \infty$.

Parameter $c(m)$:

$c(1) = 0.577215664901532..$	$c(3) = 3.132033780020806..$	$c(5) = 5.289039896592188..$	$c(7) = 7.363980242224343..$
$c(2) = 1.963510026021423..$	$c(4) = 4.227453533376265..$	$c(6) = 6.332127505374914..$	$c(8) = 8.388492663295854..$

A fast algorithm for $c(m)$ is given in (V,2).

I,4) Checking (1,3) for small k_0

In the following table, $k = T(m,n), n > 1$, is sorted and (1,2) is used for the evaluation of k_0. Then, with (1,3a): $t = 24k_1(k_1 - k_0)$.

(m,n)	k	k_0	t		(m,n)	k	k_0	t		(m,n)	k	k_0	t
(1,2)	3	2.6437	0.3129		(1,4)	30	29.153	0.8950		(5,2)	111	110.458	0.9666
(2,2)	7	6.6598	0.6193		(4,2)	43	42.740	0.9394		(1,6)	226	225.009	0.9928
(1,3)	10	9.7741	0.7399		(2,3)	56	55.627	0.9384		(6,2)	289	288.751	0.9913
(3,2)	17	16.766	0.8592		(1,5)	82	81.827	0.9663		(7,2)	762	761.413	0.9951

We see that the least occurring values of k or k_0 are large enough to satisfy (I,3a).

Even the tendency $t \to 1$ in (1,3b) is confirmed for moderately large k.

I,5) Result

(I,5a) $k_2 = k_1 - \frac{1}{24k_1} < k_0 < k_1$, with $k_1 = e^{mn-c(m)} - \frac{m+2}{2m}$, $m \geq 1, n > 1$

(I,5b) $T(m,n) = \lfloor k_0 + 1 \rfloor = \lfloor k_1 + 1 \rfloor$, if $\lfloor k_1 + 1 \rfloor = \lfloor k_2 + 1 \rfloor$.

As a conjecture, this condition generally holds. At least, it avoids bad terms.

(I,6) Remarks

A detailed discussion requires some auxiliary formulas most of which are asymptotic expansions. The derivations follow the same method: If k is large then $x = \frac{1}{k}$ is small and Taylor's formula can be applied. Numerical aspects, particularly the evaluation of $c(m)$, refer to these formulas and will be discussed in VI).

II) First asymptotic expansion

The harmonic series

$$H(k) = H_1(k-1) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} = \sum_{h=1}^{k} \frac{1}{h}$$

has the well-known asymptotic expansion

(II,2) $H(k) = \log(k) + \gamma + \frac{1}{2k} - \frac{1}{12k^2} + O\left(\frac{1}{k^4}\right)$.

We start with

$$H_m(k) = 1 + \sum_{h=1}^{k} \left(\frac{1}{mh} - f(m,h)\right) \text{ with (II,3a) } f(m,h) = \frac{1}{mh} - \frac{1}{mh+1}$$

$$= 1 + \frac{1}{m}H(k) - S(m,k) \quad \text{ with (II,3b) } S(m,k) = \sum_{h=1}^{k} f(m,h)$$

Definition (II,3a) $F(m,k) = \beta(m) - S(m,k-1)$ with $\beta(m) = \lim_{k\to\infty} S(m,k)$

An equivalent formula will be used in V,2):

(II,3b) $F(m,k) - F(m,k+1) = f(m,k)\ with \lim_{k\to\infty} F(m,k) = 0$

(II,4) $H_m(k) = 1 + \frac{1}{m}H(k) + F(m,k+1) - \beta(m)$

The expansion of $F(m,k+1)$ will be derived separately, see (V,3):

(II,5) $F(m,k+1) = \frac{1}{m^2k} - \frac{m+1}{2m^3k^2} + \frac{m^2+3m+2}{6m^4k^3} + O\left(\frac{1}{k^4}\right)$

With (II,2), (II,4) and (II,5):

$$H_m(k) = 1 + \frac{1}{m}\left(\log(k) + \gamma + \frac{1}{2k} - \frac{1}{12k^2}\right) - \beta(m) + \frac{1}{m^2k} - \frac{m+1}{2m^3k^2} + \frac{m^2+3m+2}{6m^4k^3}$$

$$H_m(k) = 1 + \frac{1}{m}\left(\log(k) + \gamma + \frac{m+2}{2mk} - \frac{m^2+6m+6}{12m^2k^2} + \frac{m^2+3m+2}{6m^3k^3}\right) - \beta(m)$$

With $c(m) = \gamma + m\big(1 - \beta(m)\big)$:

(II,6a) $H_m(k) = \frac{1}{m}\left(\log(k) + c(m) + \frac{m+2}{2mk} - \frac{m^2+6m+6}{12m^2k^2} + \frac{m^2+3m+2}{6m^3k^3}\right) + O\left(\frac{1}{k^4}\right)$ with

(II,6b) $c(m) = \gamma + m\big(1 - \beta(m)\big)$.

III) Second asymptotic expansion

(II,1) can be transformed to another asymptotic expansion which allows to invert $H_m(k)$:

(III,1) $H_m(k) = \frac{1}{m}\left(\log\left(k + \frac{m+2}{2m} + \frac{1}{24k} - \frac{m+2}{48mk^2} + O\left(\frac{1}{k^3}\right)\right) + c(m)\right).$

Proof:

Extract, with $= \frac{1}{k}$, $r(x) = \frac{m+2}{2m}x - \frac{m^2+6m+6}{12m^2}x^2 + \frac{m^2+3m+2}{6m^3}x^3$ from (II,6)

and transform the equation $log(k + g(x)) = \log(k) + r(x)$: $xg(x) = e^{r(x)} - 1$.

Taylor's formula: $xg(x) = \frac{(m+2)x}{2m} + \frac{x^2}{24} - \frac{(m+2)x^3}{48m} + O(x^4)$

$g(x) = \frac{m+2}{2m} + \frac{x}{24} - \frac{(m+2)x^2}{48m} + O(x^3) \Rightarrow g\left(\frac{1}{k}\right) = \frac{m+2}{2m} + \frac{1}{24k} - \frac{m+2}{48mk^2} + O\left(\frac{1}{k^3}\right)$ qed

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IV) Derivation of formula (I,3)

With (III,1), the equation $H_m(k) = n$ can be solved, where n is a given integer:

(IV,1) $k + \frac{m+2}{2m} + \frac{1}{24k} - \frac{m+2}{48mk^2} + O\left(\frac{1}{k^3}\right) = e^{mn-c(m)}$

A first-order approximation (neglecting $O(k^{-1})$) is $k_1 = e^{mn-c(m)} - \frac{m+2}{2m}$

Neglecting $O(k^{-3})$: $k = k_0 = k_1 - \frac{1}{24k_1} + \frac{m+2}{48m{k_1}^2} = k_1 - \frac{t}{24k_1}$ with $t = 1 - \frac{m+2}{2mk_1}$

Thus, for large k_1, we find:

(I,3a) $k_0 = k_1 - \frac{t}{24k_1}$ with $0 < t < 1$ and (I,3b) $t \to 1$ for $k_1 \to \infty$. q.e.d.

V) Series for $f(m,k), F(m,k)$ and $F(m,k+1)$

V,1) Expansion of $f(m,k) = \frac{1}{mk} - \frac{1}{mk+1}$, see (II,3a)

Transformation of $\frac{1}{mk+1}$ by a taylor series $\left(x = \frac{1}{k}\right)$: $\frac{1}{mk+1} = \frac{x}{m+x} = \frac{x}{m} \cdot \frac{1}{1+\frac{x}{m}}$

$= \frac{x}{m}\sum_{j=0}^{\infty}(-1)^j\left(\frac{x}{m}\right)^j = \frac{x}{m} + \sum_{j=2}^{\infty}(-1)^{j-1}\left(\frac{x}{m}\right)^j = \frac{1}{mk} - \sum_{j=2}^{\infty}(-1)^j\left(\frac{1}{mk}\right)^j$

Result: (V,1) $f(m,k) = \sum_{j=2}^{\infty}(-1)^j\left(\frac{1}{mk}\right)^j$

V,2) Expansion of $F(m,k)$

$$with\ F(m,k)-F(m,k+1)=f(m,k)\ and \lim_{k\to\infty} F(m,k)=0 \text{ see (II,3b)}$$

Set (V,2a) $F(m,k)=\sum_{r=1}^{\infty} a(r)\frac{1}{k^r}$

$$F(m,k)-F(m,k+1)=\sum_{r=1}^{\infty} b(r)\,D(k,r) \text{ with } D(k,r)=\frac{1}{k^r}-\frac{1}{(k+1)^r}$$

Transformations

a) $x=\frac{1}{k}$: $D(k,r)=x^r-\frac{x^r}{1+x^r}$ $h(r,x)=\frac{1}{1+x^r}=\sum_{j=0}^{\infty}(-1)^j\binom{r+j-1}{j}x^j$ leads to $D(k,r)=x^r\big(1-h(r,x)\big)$ $=x^r\sum_{j=1}^{\infty}(-1)^{j-1}\binom{r+j-1}{j}x^j$ $=x^{r+1}\sum_{j=0}^{\infty}(-1)^j\binom{r+j}{j+1}x^j$ $=\frac{1}{k^{r+1}}\sum_{j=0}^{\infty}(-1)^j\binom{r+j}{j+1}\frac{1}{k^j}$	b) $f(m,k)=\sum_{r=1}^{\infty}b(r)\,D(k,r)$ $=\sum_{r=1}^{\infty}b(r)\frac{1}{k^{r+1}}\sum_{j=0}^{\infty}(-1)^j\binom{r+j}{j+1}\frac{1}{k^j}$ $=\sum_{r=1}^{\infty}b(r)\sum_{j=0}^{\infty}(-1)^j\binom{r+j}{j+1}\frac{1}{k^{r+1+j}}$ $=\sum_{r=2}^{\infty}\sum_{j=0}^{\infty}b(r-1)(-1)^j\binom{r-1+j}{j+1}\frac{1}{k^{r+j}}$ $=\sum_{\rho=2}^{\infty}\sum_{j=0}^{\rho-2}b(\rho-j-1)(-1)^j\binom{\rho-1}{j+1}\frac{1}{k^\rho}$ and with $\rho=r+j$ $f(m,k)=\sum_{\rho=2}^{\infty}(-1)^\rho\left(\frac{1}{km}\right)^\rho$, using (V,1)

Coefficient comparison: $\sum_{j=0}^{\rho-2} b(\rho-j-1)(-1)^j\binom{\rho-1}{j+1}=\left(\frac{-1}{m}\right)^\rho$ or, with $r=\rho-1$:

(V,2b) $\sum_{j=0}^{r-1} b(r-j)(-1)^j\binom{r}{j+1}=\left(\frac{-1}{m}\right)^{r+1}$, $r=1$ *yields* $b(1)=\frac{1}{m^2}$

For $r>1$, $b(r)$ can be evaluated by $b(1), b(2), \dots, b(r-1)$:

(V,2c) $rb(r)=\left(\frac{-1}{m}\right)^{r+1}-\sum_{j=1}^{r-1} b(r-j)(-1)^j\binom{r}{j+1}$

Evaluation: $b(1)=\frac{1}{m^2}, b(2)=\frac{m-1}{2m^3}, b(3)=\frac{m^2-3m+2}{6m^4}, b(4)=-\frac{m^2-2m+1}{4m^5},\dots$

Particular result: (V,2d) $F(m,k)=\frac{b(1)}{k}+\frac{b(2)}{k^2}+\frac{b(3)}{k^3}+\cdots$

The series diverges, but is still useful, as long as the terms decrease, see annotations in VI.

V,3) Expansion of $F(m,k+1)$

Combination of (V,1) and (V,2d):

$$F(m,k+1)=F(m,k)-f(m,k)=\frac{a(1)}{k}+\frac{a(2)}{k^2}+\frac{a(3)}{k^3}-\frac{1}{m^2k^2}+\frac{1}{m^3k^3}+O\left(\frac{1}{k^4}\right)$$

(V,3) $F(m,k+1)=\frac{1}{m^2k}-\frac{m+1}{2m^3k^2}+\frac{m^2+3m+2}{6m^4k^3}+O\left(\frac{1}{k^4}\right)$

VI) The parameter c(m) and numerical aspects

In (II,7) $c(m) = \gamma + m\big(1-\beta(m)\big)$, (II,3c) $\beta(m) = \lim_{k\to\infty} S(m,k)$ has to be evaluated individually.

$m = 1$: $\beta(1) = 1$ and $c(m) = \gamma$ (Euler's constant).

Therefore $T(1,n) = \left\lfloor e^{n-\gamma} - \frac{1}{2}\right\rfloor =$ A002387$(n) - 1$.

$m = 2$: $S(2,k) = \sum_{h=1}^{k}\left(\frac{1}{2h} - \frac{1}{2h+1}\right)$, $\beta(2) = \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} \pm \cdots = 1 - \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \pm \cdots\right)$

$\beta(2) = 1 - \log(2) \Rightarrow c(2) = \gamma + 2\log(2) = \gamma + \log(4)$,

$T(2,n) = \left\lfloor e^{2n-\gamma-\log(4)}\right\rfloor = \lfloor \tfrac{1}{4} e^{2n-\gamma}\rfloor =$ A092315(n).

For $m > 2$, $\beta(m)$ has to be evaluated numerically. We write (II,3c) as

(VI,1a) $\beta(m) = \sum_{r=1}^{k-1} f(m,r) + F(m,k)$

with $f(m,r) = \frac{1}{mr(mr+1)}$ and $F(m,k) = \sum_{r=k}^{\infty} f(m,r)$

Rough estimate: (VI,1b) $F(m,k) \approx \frac{1}{m^2}\sum_{r=k}^{\infty}\frac{1}{r^2} \approx \frac{1}{m^2}\int_k^{\infty}\frac{dr}{r^2} = \frac{1}{m^2 k}$

The precision p digits, i.e. $F(m,k) < \varepsilon = 10^{-p}$ (maximum error) requires $k > \frac{1}{m^2\varepsilon} = \frac{10^{100}}{m^2}$ for $p = 100$. No computer can add that many terms in $\sum_{r=1}^{k-1} f(m,r)$. But we can fix this problem by using (V,2a) $F(m,k) = \sum_{r=1}^{\infty} b(r)\frac{1}{k^r}$,

where the coefficients $b(r)$ are defined by the recurrence

(V,2c) $b(r) = \frac{1}{r}\left(\left(\frac{-1}{m}\right)^{r+1} - \sum_{j=1}^{r-1} b(r-j)(-1)^j\binom{r}{j+1}\right)$ with $b(1) = \frac{1}{m^2}$

Note that $b(1)/k$ is the rough estimate (VI,1b).

With $c(m) = \gamma + m\big(1-\beta(m)\big)$, the algorithm is complete.

Annotations:

The formula (V,2a) $F(m,k) = \sum_{r=1}^{\infty} b(r)\frac{1}{k^r}$ is, strictly speaking, not correct because the series is divergent: $d(r) = b(r)\frac{1}{k^r} \to \pm\infty$. But as long as $|d(r)|$ is decreasing, $d(r)$ approximately indicates the deviation from $\beta(m)$ in (VI,1a). For example, if the deviation is to be smaller than 10^{-100}, $k = 100$ can be used: $d(72) = 3\cdot 10^{-100}$, $d(73) = 2\cdot 10^{-101}$, for $m = 3$. In this case, $F(m,k)$ begins:

$$F(m,k) = \frac{1}{9}10^{-2} + \frac{1}{27}10^{-4} + \frac{1}{243}10^{-6} - \frac{1}{243}10^{-8} \ldots$$

A numerical problem remains. Let d be the difference of $k_1 = e^{mn-c(m)} - \frac{m+2}{2m}$ and the next integer, i.e. $d = min(k_1 - \lfloor k_1\rfloor, 1 + \lfloor k_1\rfloor - k_1)$. Normally (always?), the uncertainty $\frac{1}{24k_1}$ of k_0, see (I,3a), is very small compared with d so that we can neglect it. But a numerical error ε in $c(m)$ induces an error δ in $k_1 = e^{mn-c(m)} - \frac{m+2}{2m}$:

$$\delta = e^{mn-c(m)} - e^{mn-c(m)-\varepsilon} \approx \varepsilon k_1.$$

This requires $\delta < d$ or $k_1 < \frac{d}{\varepsilon}$. For example, $\varepsilon = 10^{-100}, d = 0.1 \Rightarrow k_1 < 10^{99}$.