Fibonacci-Like Sequences and their Cassini Constants, Multiplication and Factorisation.

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Foreword

I graduated in Mathematics long ago, but worked as a Systems Programmer so I don't claim to be a mathematician. In an idle moment I pondered on a proof of the convergence of the Fibonacci sequence, and realised it applies to any Fibonacci-like sequence and reveals a constant for the sequence. Then I followed a trail of discovery, more or less as detailed in this document.

This document uses little more than High School mathematics, and is liberally illustrated with examples. I may not have used the received terminology, and have taken the name of Cassini for the constant since it plugs straight into Cassini's Identity. My notation uses (r,s) for a sequence in terms of its root values rather than its starting values. The final conjectures, while easy to understand, may be challenging to prove! Or maybe they have been?

Everything in this document I have discovered independently, but subsequent searches show that many of the results are already documented in one form or another. In particular the comments in OEIS® Sequence A082970 by Wolfdieter Lang precede many of mine. Also by T.D.Noe , Shreevatsa R , Matthew Staller and Robert Israel ; and a reference to a paper by Alfred Brousseau in 1963, who suggests ordering Fibonacci-like sequences in exactly the order I chose. Nevertheless, I hope to have made some contribution.

Some supporting material, perl programs and output, can be found at:

https://www.dropbox.com/sh/jwju9byfibiu41e/AAClhxJlAcc9vhd7vnLnX1nXa?dl=0

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Summary

A simple proof of the convergence of a general Fibonacci-like sequence leads to the **Cassini constant, K**, which we then use to generalise some identities. Extending such an integer sequence backwards reveals a **unique origin** about which the sequence can be reflected, giving two **complementary sequences**.

These give us **root values** which define how a sequence is derived from the Fibonacci sequence. The asymptotic ratio of the terms of a general sequence to the Fibonacci sequence gives **skewed-roots** of the Cassini constant. We use these to create a **formula for the nth term** of any Fibonacci-like sequence.

Further derivations, using the root values of one sequence as multipliers in a simple linear combination of another sequence, is a process which has direct parallels with multiplication, with the Fibonacci sequence acting as an identity. The corresponding results for **sequence multiplication** are listed and proved.

In particular: the Cassini constant of the product of sequences is the product of the Cassini constants of the sequences. Further, the product of two complementary sequences is a simple multiple of the Fibonacci sequence.

We show how to find the roots of the product of two sequences.

For **sequence factorisation** we make some number theoretic conjectures about Cassini constants, based on observations and strong empirical evidence. The foremost of these are that a prime Cassini constant results from just one pair of complementary **prime sequences**, and non-prime Cassini constants have prime factors which are themselves all Cassini constants.

More conjectures are made and we show how the factors of Cassini constants can be used to factorise a sequence whose roots are known, or to determine how many, and which, sequences give rise to a given Cassini constant, K. This leads to the integer solutions r and s of the equation: $(r^2 + 3rs + s^2) = K$.

Finally, we introduce OEIS® sequence A336403 and show it is closed under multiplication, filling a gap in sequences related to the Cassini constants.

Fibonacci-Like Sequences

A **Fibonacci-like sequence** is one whose terms are created by adding the two previous terms, i.e. S(i+2)=S(i+1)+S(i), and is defined by two **starting values a,b** conventionally given as S(0) and S(1). For example:

a=0, b=1 gives: 0,1,1,2,3,5,8 … the Fibonacci Sequence F;
a=2, b=1 gives: 2,1,3,4,7,11,18 … the Lucas Sequence L;
a=4, b=5 gives: 4,5,9,14,23,37,60 … a general Fibonacci-like sequence S.

Starting with terms S(i)=u, S(i+1)=v, we get S(i+2)=(u+v), S(i+3)=(u+2v), S(i+4)=(2u+3v), and it becomes obvious that the coefficients are consecutive terms from the Fibonacci sequence itself. The general term can be written as **S(i+j) =** F(j−1)u + F(j)v = **F(j−1)S(i) + F(j)S(i+1)** or, by replacing i with i−1, more neatly as: S(i+j−1) = F(j−1)S(i−1) + F(j)S(i) or **S(i+j−1) = F(i−1)S(j−1) + F(i)S(j)** .

Note that in this expression the index origin of the Fibonacci Sequence must be fixed at F(0)=0 to get the required coefficients. References to the Fibonacci and Lucas Sequences will always use the time-honoured origins F(0)=0 and L(0)=2.

The general sequence S can, for the moment, have a more flexible origin.
For example, if 4=S(1) and 5=S(2) in the general example above, we get:
S(7) = S(2+5) = F(4)S(2) + F(5)S(3) = 3×5 + 5×9 = 15 + 45 = 60,
or S(7) = S(3+5−1) = F(2)S(4) + F(3)S(5) = 1×14 + 2×23 = 14 + 46 = 60.

If we know the values of any two terms S(i) and S(i+j) and their separation 'j', the expression above will give us S(i+1) and the sequence is fully determined; but in general any two given separated values may not define an integer sequence, for example 0,x,y,3 would give x=y=3/2 .

Later, we shall be interested in sequences derived from an *alternate* pair r,,s. Here the intermediate term *is* an integer so generates an integer sequence.

Simple Identity

For any Fibonacci-like sequence S (including F and L):
S(n+r) + (−1)^{r}S(n−r) = L(r)S(n)
where L is the Lucas Sequence with L(0)=2 and L(1)=1.

Proof:
For r=0: S(n) + S(n) = 2S(n) = L(0)S(n)
For r=1: S(n+1) − S(n−1) = S(n) = L(1)S(n)
For r+2: S(n+(r+2)) + (−1)$^{(r+2)}$S(n−(r+2))

$= S(n+(r+1)) + S(n+r) + (-1)^r(S(n-r) - S(n-(r+1)))$
$= S(n+r) + (-1)^r S(n-r) + S(n+(r+1)) + (-1)^{(r+1)} S(n-(r+1))$
$= L(r)S(n) + L(r+1)S(n) = L(r+2)S(n)$ [QED]

For example: $S(n+2) + S(n-2) = 3S(n)$, where 3=L(2).

When r=n and S is the Fibonacci sequence F: $F(2n) = L(n)F(n)$.
When r=n and S is the Lucas sequence L: $L(2n) = (L(n))^2 \pm 2$ (as n is odd or even).

Convergence

For any Fibonacci-like sequence S, the ratio S(i+1)/S(i) converges to the Golden Ratio $\Phi \approx 1.618034\ldots$, being the positive solution of the equation $x^2 = x + 1$.

The equation $\Phi^2 = \Phi + 1$ can also be written as $\Phi = 1 + (1/\Phi)$.
Let u, v, (u+v), (u+2v) be any four consecutive terms in the sequence (≠0).
If **$(v/u) < \Phi$** then $(u/v) > (1/\Phi)$, so that **$(u+v)/v$** $= (u/v) + 1$ **$>$** $(1/\Phi) + 1 =$ **Φ** ; and vice versa. I.e. the ratio of consecutive terms alternates either side of Φ.

To show convergence we look at the difference between consecutive ratios.
The first difference is $(u+v)/v - v/u =$ **$(u(u+v) - v^2)/uv$** .
The next difference is $(u+2v)/(u+v) - (u+v)/v = (v(u+2v) - (u+v)^2)/v(u+v)$
$= (uv + 2v^2 - u^2 - 2uv - v^2)/v(u+v) = (v^2 - u^2 - uv)/v(u+v) =$ **$((v^2 - u(u+v))/v(u+v)$**.

From which we see that the numerator is constant with alternating sign, but the denominator increases as the sequence progresses, proving convergence.

The Cassini Constant

We can express the above constant in terms of the starting values **a,b** of S as **$(a(a+b) - b^2)$**, and call it the **Cassini constant: K, K_S or $K_{a,b}$**.

a=0, b=1 gives **$K_F=K_{0,1}=-1$** for the Fibonacci Sequence (but **$K_{1,0}=+1$**),
a=2, b=1 gives **$K_L=K_{2,1}=+5$** for the Lucas Sequence,
a=4, b=5 gives **$K_S=K_{4,5}=11$** for the general example above.

If S(0)=a and S(1)=b then $K_S=K_{a,b}=S(0)S(2)-S(1)^2$ and we showed above that, for any n: **$S(n-1)S(n+1) - S(n)^2 = (-1)^{n+1}K_S = (-1)^n(-K_S)$**. This is a generalisation of **Cassini's Identity** for the Fibonacci sequence, where $K_F=-1$. Later we shall fix the index origin and generalise some other identities.

In the above, a,b can be any real numbers. If (b/a)=Φ, then every pair has the same ratio Φ, and **K=0**. Going backwards down this sequence the terms get smaller and approach 0, but always remain positive.

From now on, we shall be dealing only with *integer* sequences.

Scalar Multiples

If any two consecutive terms of a sequence (or indeed any alternate pair) have a common factor λ, then every term in the sequence has the same factor λ. Then define T such that T(i)=S(i)/λ. We say that S is a **scalar multiple** of T and we write **S=λT**. (When using an index we ignore the semantic ambiguity of "λT(n)", since the n^{th} term of λT has the value λ×T(n) by definition).

Clearly, the Cassini constant is $\mathbf{K_{\lambda T}=\lambda^2 K_T}$.

Since every integer divides zero, any sequence with a zero term has the next term as a common factor, λ say, and the sequence contains …,−λ,λ,0,λ,λ,…
In this case, S is a scalar multiple of the Fibonacci sequence, i.e. **S=λF**.

Backward Extensions and Negative Indices and Terms

Any Fibonacci-like sequence can be extended backwards into negative indices using: S(i−1) = S(i+1) − S(i). Any integer sequence extending backwards will include negative terms: positive terms become smaller until S(i+1) < (S(i+2))/2 , whence S(i) > S(i+1). Extending back beyond this point makes S(i−1) negative, and then gives alternating negative and positive terms.
E.g. … −8,5,−3,2,−1,1,0,1,1,2,3,5 … for the Fibonacci sequence,
and … −11,7,−4,3,−1,2,1,3,4,7,11,18 … for the Lucas sequence,
and … −7,5,−2,3,1,4,5,9,14,23 … for the sequence defined by a=4, b=5.

Apart from its sign, the Cassini constant can be calculated from any two consecutive values anywhere in an extended sequence.

Sequence/Index Origin

Since a sequence is endless in both directions we need to define the origin.

If a sequence contains a zero value it is either the Fibonacci sequence F or a scalar multiple λF, so we use the zero value as the origin. The starting values are λF(0)=0, λF(1)=λ, and the Cassini constant is negative: $K_{0,\lambda}=-\lambda^2$.

From now on we refer to F or λF specifically, and use other symbols such as S for sequences with no zero value or which have a different index origin.

For the general sequence S we choose the origin S(0) so that S(−1) is the "rightmost" negative term. This gives the following properties:

S(−1) and S(1) are the *only* alternate pair with opposite signs;
S(0) and S(1) are the *first* consecutive pair of positive numbers;
S(0) and S(1) are the *only* descending positive pair, i.e. S(0) > S(1);
S(−1) and S(1) have the *smallest sum* of absolute values of any alternate pair;
for all i≥0 we have S(i)>0; and for all i<0 we have S(2i)>0 and S(2i+1)<0.

Let $^{a}\mathbf{S}_{b}$ denote a sequence in term of its starting values S(0)=a and S(1)=b.
Since a>b, we have $a(a+b)>2b^2$, so $K_S>b^2$ and is *always positive*.

To show the origin of a sequence, we now show it in square brackets:
e.g. … −8,5,−3,2,−1,1,**[0]**,1,1,2,3,5 … for the Fibonacci sequence F,
and … −11,7,−4,3,−1,**[2]**,1,3,4,7,11 … for the Lucas sequence L,
which still has the conventional origin and index so we can write $\mathbf{L} = {}^{2}\mathbf{S}_{1}$,
and … −19,12,−7,5,−2,**[3]**,1,4,5,9,14 … for $^{3}\mathbf{S}_{1}$, now starting with a=3, b=1.

Generalising Some Identities

Having defined an index origin above, we can now generalise some identities.
We have proved the first, but the rest are stated without proof.

Cassini's Identity: $F(n-1)F(n+1) - F(n)^2 = (-1)^n$
becomes $S(n-1)S(n+1) - S(n)^2 = (-1)^n(-K_S)$

Catalan's Identity: $F(n)^2 - F(n-r)F(n+r) = (-1)^{n-r}F(r)^2$
becomes $S(n)^2 - S(n-r)S(n+r) = (-1)^{n-r}F(r)^2(-K_S)$

Vajda's Identity: $F(n+i)F(n+j) - F(n)F(n+i+j) = (-1)^nF(i)F(j)$
becomes $S(n+i)S(n+j) - S(n)S(n+i+j) = (-1)^nF(i)F(j)(-K_S)$

d'Ocagne's Identity: $F(m)F(n+1) - F(n)F(m+1) = (-1)^nF(m-n)$
becomes $S(m)S(n+1) - S(n)S(m+1) = (-1)^nF(m-n)(-K_S)$

We can make these pairs homologous by appending the term "$(-K_F)$" to the right-hand side of the base identities, since $(-K_F)=1$.

Complementary Sequences

Simple observation shows that the example sequence $S={}^{3}S_{1}$
S = … 12,−7,5,−2,[3],1,4,5,9 …
can be reversed, with appropriate sign changes, to give $R={}^{3}S_{2}$
R = … 9,−5,4,−1,[3],2,5,7,12 …

For any sequence $\mathbf{S}={}^{a}\mathbf{S}_{b}$ (S(i)≠0) we define the **complementary sequence S'** as follows: for all i: $\mathbf{S'(i)=(-1)^{i}S(-i)}$, which has the effect of reversing/reflecting S about S(0), with the appropriate sign changes. The complement of S’ is S again, i.e. (S’)’=S. The complement of a scalar multiple is the scalar multiple of the complement, i.e. (λS)’=λ(S’). In general, the complement of $^{a}\mathbf{S}_{b}$ is $^{a}\mathbf{S}_{a-b}$ and vice versa. Therefore the complement of the Lucas sequence is itself: L'=$^{2}S_{1}$=L.

S’(0)=S(0), S’(−1)=−S(1)<0 and S’(1)=−S(−1)>0, so the index origin is unchanged. Substituting S’ for S and n=0 in Cassini's identity, sign changes cancel out and **$K_{S'}$=K_S**. I.e. **complementary sequences have the same Cassini constant.**

Substituting $(-1)^{i}S(-i)$ in the Simple Identity with n=0 we get S(i) + S’(i) = S(0)L(i) or **S+S’=S(0)L, the complements identity:** the term-by-term sum of a sequence plus its complement is the S(0)-times scalar multiple of the Lucas sequence.

If S’(1)=S(1)=λ then S’=S=λL, otherwise we nominate the smaller of S(1) and S’(1) as the Primary sequence and the larger as the Secondary sequence.
With starting values [a],b the Primary sequence has b<(a−b) i.e. b<a/2.

Root Terms and Values

A sequence S can be defined by any alternate pair of terms. We choose the *only* alternate pair with opposite signs, S(−1)<0 and S(1)>0, as **root terms**.
We define **root values: r = −S(−1) and s = S(1)**, with r and s *both positive*, and r+s=S(0). The complementary sequence has root values **s** and **r**.
Sometimes we simply refer to the roots, where the context is unambiguous.

Cassini's identity with n=0 gives K in terms of r and s:
the Cassini constant **$K = (r+s)^2 + rs = r^2 + 3rs + s^2$** .
We use the notation $_{r}\mathbf{S}_{s}$ or just **(r,s)** for the sequence and $_{r}\mathbf{K}_{s}$ for the constant.

In terms of starting values a,b we have a=S(0)=r+s and b=S(1)=s, so:
$_{r}\mathbf{S}_{s} = {}^{r+s}\mathbf{S}_{s}$ with complement $_{s}\mathbf{S}_{r} = {}^{r+s}\mathbf{S}_{r}$ and $_{r}\mathbf{K}_{s} = {}_{s}\mathbf{K}_{r} = \mathbf{K}_{r+s,s}$.

Two numbers with no common factor are said to be **relatively prime**.
Clearly, root values are relatively prime if and only if the starting values are.

Complements and Root Terms of the Anomalous Fibonacci Sequence

The Fibonacci sequence clearly reflects about its zero value, which makes the natural origin F(0)=0; but the negative values fall on *even* negative indices. For the complement F’ to be F the definition would have to be **$F'(i)=(-1)^{i\pm1}F(-i)$**; but then F(i)+F’(i)=2F(i)=L(0)F(i) and not F(0)L(i), so the complements identity fails.

Using F(−1)=1 and F(1)=1 as root terms gives root values −1,1 and makes K_F=−1 negative, as we have already seen in the identities above. Further, since F(−1) is *positive*, the index differs from our standard definition of origin described earlier, and F fails many of the properties associated with it.

The problems seem to stem from the zero value, which is neither positive nor negative. Consider a general sequence S with the *same values* as the Fibonacci sequence. If we choose the origin such that S(−1) and S(1) are *negative* and *non-negative* we get S = $_1S_0$: …,−1,[1],0,…; and if we base the origin on S(−1) and S(1) being *non-positive* and *positive* we get S′ = $_0S_1$: …,0,[1],1,….
These two sequences are complementary with the normal definition.

Note that 0 = S(1) = F(0) = S′(−1) and in general S(i+1) = F(i) = S′(i−1). The complements identity applies, so S(i) + S′(i) = S(0)L(i) = L(i) , and we have come in a roundabout way to the well known identity: **F(i−1) + F(i+1) = L(i)**, i.e. the sum of alternate terms of the Fibonacci sequence gives the Lucas sequence.

The index origin F(0)=0 is required for the identities listed earlier, but in much of what follows we use the sequence (1,0) or its complement (0,1) which have the same values, and non-negative roots so that $K_{1,0} = K_{0,1} = +1$.

The On-line Encyclopedia of Integer Sequences®, https://oeis.org/

The set of all (positive) Cassini constants is a known sequence listed in OEIS® as A031363. The subset of Cassini constants created by relatively prime starting values is also listed, as A089270; and the further subset of Cassini constants which are prime numbers is listed as A038872.

Below are extracts from their entries in OEIS®, and some comments.

A031363:

Positive numbers of the form x^2 + xy - y^2; or, of the form 5x^2 - y^2.

1, 4, 5, 9, 11, 16, 19, 20, 25, 29, 31, 36, 41, 44, 45, 49, 55, 59, 61, 64, 71, 76, 79, 80, 81, 89, 95, 99, 100, 101, 109, 116, 121, 124, 125, 131, 139, 144, 145, 149, 151, 155, 164, 169, 171, 176, 179, 180, 181, 191, 196, 199, 205, 209, 211, 220, 225, 229, 236

The first form is x(x+y) – y^2 so these are all the constants for Fibonacci-like sequences, including scalar multiples, with starting values [x],y.
Later, we show that this set is closed under multiplication.

A089270:

Positive numbers represented by the integer binary quadratic form x^2 + x*y - y^2 with x and y relatively prime.

1, 5, 11, 19, 29, 31, 41, 55, 59, 61, 71, 79, 89, 95, 101, 109, 121, 131, 139, 145, 149, 151, 155, 179, 181, 191, 199, 205, 209, 211, 229, 239, 241, 251, 269, 271, 281, 295, 305, 311, 319, 331, 341, 349, 355, 359, 361, 379, 389, 395, 401, 409, 419, 421, 431

I.e. all the constants for relatively prime starting values. This set includes 1.

Any sequence S can be expressed as λ(r,s) where r and s are relatively prime. Since $K_S = \lambda^2({}_rK_s)$ there is a natural one-to-one mapping between A031363 and the Cartesian product of A000290 (the squares) and A089270.
A089270 itself also contains some squares, e.g. 121 = ${}_7K_3$ and 361 = ${}_9K_8$.

A038872:

Primes congruent to {0, 1, 4} mod 5.

5, 11, 19, 29, 31, 41, 59, 61, 71, 79, 89, 101, 109, 131, 139, 149, 151, 179, 181, 191, 199, 211, 229, 239, 241, 251, 269, 271, 281, 311, 331, 349, 359, 379, 389, 401, 409, 419, 421, 431, 439, 449, 461, 479, 491, 499, 509, 521, 541, 569, 571, 599, 601, 619

Primes of the form $x^2 + xy - y^2$ (as well as of the form $x^2 + 3xy + y^2$).

I.e. all the constants which are prime numbers. This set excludes 1.
Only 5 can be 0 mod 5 and prime; while 1 and 4 are ±1 mod 5 and must be odd, so are ±1 mod 10. Hence, after 5, all the prime constants end in 1 or 9.

A045468:

Primes congruent to {1, 4} mod 5.

11, 19, 29, 31, 41, 59, 61, 71, 79, 89, 101, 109, 131, 139, 149, 151, 179, 181, 191, 199, 211, 229, 239, 241, 251, 269, 271, 281, 311, 331, 349, 359, 379, 389, 401, 409, 419, 421, 431, 439, 449, 461, 479, 491

This is clearly A038872 without the 5 term.

A336403:

The subsequence of A089270 which excludes terms divisible by 5.

1, 11, 19, 29, 31, 41, 59, 61, 71, 79, 89, 101, 109, 121, 131, 139, 149, 151, 179, 181, 191, 199, 209, 211, 229, 239, 241, 251, 269, 271, 281, 311, 319, 331, 341, 349, 359, 361, 379, 389, 401, 409, 419, 421, 431, 439, 449, 451, 461, 479, 491, 499, 509

Later, we show that this set is closed under multiplication.

Examples of Sequences and their Cassini Constants

In the next part of this document we shall look at the relationships between sequences, and the significance of the factors of their Cassini constants.

We now show more example sequences to inform our subsequent discussion.

We list them in the order of their starting values **[a],b** with **a** the major number. We do not show scalar multiples and we restrict **b** to not more than **a/2**, to show only Primary sequences (in which the Secondary can be seen backwards, with sign changes). The Fibonacci sequence is shown as [1],0.

We show a few terms either side of the origin to start the sequence and its complement. We also show the Cassini constant K and its factors.
Note that K is in ascending order *only* within each major starting value [a].

... 5 -3 2 -1 [1] 0 1 1 2 ... K=1
... 7 -4 3 -1 [2] 1 3 4 7 ... K=5 (prime)
... 12 -7 5 -2 [3] 1 4 5 9 ... K=11 (prime)
... 17 -10 7 -3 [4] 1 5 6 11 ... K=19 (prime)
... 22 -13 9 -4 [5] 1 6 7 13 ... K=29 (prime)
... 19 -11 8 -3 [5] 2 7 9 16 ... K=31 (prime)
... 27 -16 11 -5 [6] 1 7 8 15 ... K=41 (prime)
... 32 -19 13 -6 [7] 1 8 9 17 ... K=55 = 5 x 11
... 29 -17 12 -5 [7] 2 9 11 20 ... K=59 (prime)
... 26 -15 11 -4 [7] 3 10 13 23 ... K=61 (prime)
... 37 -22 15 -7 [8] 1 9 10 19 ... K=71 (prime)
... 31 -18 13 -5 [8] 3 11 14 25 ... K=79 (prime)
... 42 -25 17 -8 [9] 1 10 11 21 ... K=89 (prime)
... 39 -23 16 -7 [9] 2 11 13 24 ... K=95 = 5 x 19
... 33 -19 14 -5 [9] 4 13 17 30 ... K=101 (prime)
... 47 -28 19 -9 [10] 1 11 12 23 ... K=109 (prime)
... 41 -24 17 -7 [10] 3 13 16 29 ... K=121 = 11 x 11
... 52 -31 21 -10 [11] 1 12 13 25 ... K=131 (prime)
... 49 -29 20 -9 [11] 2 13 15 28 ... K=139 (prime)
... 46 -27 19 -8 [11] 3 14 17 31 ... K=145 = 5 x 29
... 43 -25 18 -7 [11] 4 15 19 34 ... K=149 (prime)
... 40 -23 17 -6 [11] 5 16 21 37 ... K=151 (prime)
... 57 -34 23 -11 [12] 1 13 14 27 ... K=155 = 5 x 31
... 45 -26 19 -7 [12] 5 17 22 39 ... K=179 (prime)
... 62 -37 25 -12 [13] 1 14 15 29 ... K=181 (prime)
... 59 -35 24 -11 [13] 2 15 17 32 ... K=191 (prime)
... 56 -33 23 -10 [13] 3 16 19 35 ... K=199 (prime)
... 53 -31 22 -9 [13] 4 17 21 38 ... K=205 = 5 x 41
... 50 -29 21 -8 [13] 5 18 23 41 ... K=209 = 11 x 19
... 47 -27 20 -7 [13] 6 19 25 44 ... K=211 (prime)
... 67 -40 27 -13 [14] 1 15 16 31 ... K=209 = 11 x 19
... 61 -36 25 -11 [14] 3 17 20 37 ... K=229 (prime)
... 55 -32 23 -9 [14] 5 19 24 43 ... K=241 (prime)
... 72 -43 29 -14 [15] 1 16 17 33 ... K=239 (prime)
... 69 -41 28 -13 [15] 2 17 19 36 ... K=251 (prime)
... 63 -37 26 -11 [15] 4 19 23 42 ... K=269 (prime)
... 54 -31 23 -8 [15] 7 22 29 51 ... K=281 (prime)
... 77 -46 31 -15 [16] 1 17 18 35 ... K=271 (prime)
... 71 -42 29 -13 [16] 3 19 22 41 ... K=295 = 5 x 59
... 65 -38 27 -11 [16] 5 21 26 47 ... K=311 (prime)
... 59 -34 25 -9 [16] 7 23 30 53 ... K=319 = 11 x 29
... 82 -49 33 -16 [17] 1 18 19 37 ... K=305 = 5 x 61
... 79 -47 32 -15 [17] 2 19 21 40 ... K=319 = 11 x 29
... 76 -45 31 -14 [17] 3 20 23 43 ... K=331 (prime)
... 73 -43 30 -13 [17] 4 21 25 46 ... K=341 = 11 x 31
... 70 -41 29 -12 [17] 5 22 27 49 ... K=349 (prime)
... 67 -39 28 -11 [17] 6 23 29 52 ... K=355 = 5 x 71
... 64 -37 27 -10 [17] 7 24 31 55 ... K=359 (prime)
... 61 -35 26 -9 [17] 8 25 33 58 ... K=361 = 19 x 19
... 87 -52 35 -17 [18] 1 19 20 39 ... K=341 = 11 x 31
... 75 -44 31 -13 [18] 5 23 28 51 ... K=389 (prime)
... 69 -40 29 -11 [18] 7 25 32 57 ... K=401 (prime)
... 92 -55 37 -18 [19] 1 20 21 41 ... K=379 (prime)
... 89 -53 36 -17 [19] 2 21 23 44 ... K=395 = 5 x 79
... 86 -51 35 -16 [19] 3 22 25 47 ... K=409 (prime)
... 83 -49 34 -15 [19] 4 23 27 50 ... K=421 (prime)
... 80 -47 33 -14 [19] 5 24 29 53 ... K=431 (prime)
... 77 -45 32 -13 [19] 6 25 31 56 ... K=439 (prime)
... 74 -43 31 -12 [19] 7 26 33 59 ... K=445 = 5 x 89
... 71 -41 30 -11 [19] 8 27 35 62 ... K=449 (prime)
... 68 -39 29 -10 [19] 9 28 37 65 ... K=451 = 11 x 41
... 97 -58 39 -19 [20] 1 21 22 43 ... K=419 (prime)
... 91 -54 37 -17 [20] 3 23 26 49 ... K=451 = 11 x 41
... 79 -46 33 -13 [20] 7 27 34 61 ... K=491 (prime)

This list is continued in Appendix A.

Derived Sequences

We showed earlier how the Lucas sequence can be derived from the Fibonacci sequence by adding alternate terms of F, i.e. L(i) = F(i−1) + F(i+1) .
If this process is repeated on the Lucas sequence the result is 5F, the 5-times scalar multiple of Fibonacci, i.e. L(i−1) + L(i+1) = 5F(i) , as shown in the table:

Fibonacci:	−8	5	−3	2	−1	1	[0]	1	1	2	3	5	8	$K_{1,0}$=1
Lucas =$_1S_1$:	18	−11	7	−4	3	−1	[2]	1	3	4	7	11	18	K_L=5
5F:	…	25	−15	10	−5	5	[0]	5	5	10	15	25	…	$K_{5,0}$=25

The index origins line up in the table (only) because F and 5F have a different rule to L, and we can use index "i" on both sides of the derivation definitions. Note that for λF we show the positive Cassini constants $K_{\lambda,0}$.

A similar derivation using the *root values* r,s of a sequence as respective multipliers, i.e. S(i) = **r**F(i−1) + **s**F(i+1) , actually results in the sequence $_r\mathbf{S}_s$:

Fibonacci:	−3	2	−1	1	[0]	1	1	2	3	$K_{1,0}$=1
(r,s)=$_rS_s$:	5r+2s	−(3r+s)	2r+s	−r	[r+s]	s	r+2s	r+3s	2r+5s	K=$_rK_s$

The roots r,s can represent either the Primary or the Secondary sequence.

Sequence Ratios and Skewed-Roots of K

Earlier we showed that the ratio of consecutive terms in any Fibonacci-like sequence converges to Φ, with K being a measure of how quickly. Since the index origins line up (for the first derivation) we now consider the asymptotic ratio between terms of a general sequence and the Fibonacci sequence.

Three consecutive terms of F converge to u/Φ,u,uΦ and since 1/Φ=Φ−1 these are u(Φ−1),u,uΦ. The corresponding middle term in $_rS_s$ derived by using the roots r,s becomes ru(Φ−1)+suΦ, giving a ratio of (r+s)Φ−r between $_rS_s$ and F, i.e. **$_rS_s$(n)→((r+s)Φ−r)F(n)** as n→∞.

The complement sequence $_sS_r$ has a ratio to F of (r+s)Φ−s.

We call **(r+s)Φ−r** and **(r+s)Φ−s** the **skewed-roots of $_rK_s$**.
Note that skewed-roots are with respect to a specific sequence.

These roots have the following properties: they
(i) differ by the integer **|r−s|**,
(ii) sum to **(r+s)/Φ**: (r+s)Φ−r + (r+s)Φ−s = (r+s)Φ−(r+s) = (r+s)(Φ−1) = (r+s)/Φ,

(iii) multiply to $_r\mathbf{K}_s$ (hence their name):
$((r+s)\Phi-r)((r+s)\Phi-s) = (r+s)^2\Phi^2-(r+s)\Phi(r+s)+rs = (r+s)^2(\Phi^2-\Phi)+rs = \mathbf{(r+s)^2+rs}$.

If r>s the first skewed-root is less than $\sqrt{_rK_s}$ and the second is greater; and they are the ratios of the Primary and Secondary sequences to F, respectively.
The ratio of the Primary sequence to the Secondary sequence is:
$_rS_s(n)/_sS_r(n) \to ((r+s)\Phi-r)/((r+s)\Phi-s)$ or $((r+s)\Phi-r)^2/_rK_s$, as $n\to\infty$.

Examples (to 3 decimal places):

$_rS_s$:	$_1S_1$=L	$_2S_1$	$_3S_1$	$_4S_1$	$_3S_2$	...	$_8S_5$	$_{13}S_1$
$(r+s)\Phi-r$:	2.236	2.854	3.472	4.090	5.090	...	13.034	9.652
$(r+s)\Phi-s$:	2.236	3.854	5.472	7.090	6.090	...	16.034	21.652
$\prod={_rK_s}$:	5	11	19	29	31	...	209	209

For K_L both skewed-roots are $\sqrt{5}$, i.e. $L(n)\to \sqrt{5}\times F(n)$ as $n\to\infty$.

The products 29 and 31 have all four skewed-roots based on 5Φ = 8.090...

Since 209 is the Cassini constant of two different sequences, it has different and unrelated pairs of skewed-roots with respect to each sequence.

General (Asymptotic) Expression For S(n)

It is well known (e.g. Binet's formula) that $F(n)\to\Phi^n/\sqrt{5}$ and $L(n)\to\Phi^n$.
We now also have $_rS_s(n)\to((r+s)\Phi-r)F(n)$, so we can combine to give:

General Formula: $\mathbf{_rS_s(n)\to((r+s)\Phi-r)\Phi^n/\sqrt{5}}$ **as** $\mathbf{n\to\infty}$**.**

E.g. for $_8S_5$: $((r+s)\Phi-r)/\sqrt{5} \approx 13.034/2.236$ (from the table above) ≈ 5.829.
So $_8S_5(10) \approx 5.829 \times 1.618^{10} \approx 5.829 \times 122.966 \approx 716.767 \approx$ integer 717 (✓).
In this example, we have rounded to 3 decimal places at every stage, and still get the correct result when rounding to the nearest integer. Using an 8-digit calculator gives $_8S_5(10)$=716.94170, and correct rounded results for $_8S_5(n)$ whenever n≥6. A rule of thumb for $_rS_s(n)$ may be accuracy when n≥(r+s)/2?

Multiplication of Sequences

We can take any sequence P=(p,q) and perform a derivation using the roots r,s of S=(r,s) as multipliers to produce a sequence with terms: rP(i−1) + sP(i+1) :

(p,q):	(2p+q)	−p	[p+q]	q	p+2q
(p,q)(r,s):	...	2pr+qr+ps+qs	−pr+qs	pr+qr+ps+2qs	...

We call this **multiplication** of sequences and, depending on context, use the notations PS, P×(r,s), P(r,s) or (p,q)(r,s) to represent this process and the result, which we call the **product** of P and S; and use power notation e.g. P^2 for P×P.

If P above is F, i.e. p=−1 and q=1, we get the results … −r , [r+s] , s … , so **FS=S**.
If S above is F, e.g. r=−1 and s=1, we get the results … −p , [p+q], q … , so **PF=P**.
Hence **F is the unit element in sequence multiplication**.

We could equally use roots 1,0 or 0,1 for F, since the index origin of a product is determined by the resulting values. In general, the origin cannot be labelled in the table above, but one exception is for complementary sequences:

The Product of Complementary Sequences

Suppose (p,q) above is the complement (s,r). Substituting p=s and q=r, the three derived terms shown become: $r^2 + 3rs + s^2$, [0] , $r^2 + 3rs + s^2$, so
the product of $_rS_s$ and its complement $_sS_r$ is a scalar multiple: SS' = ($_rK_s$)F.

Since the Lucas sequence is its own complement, **L^2=5F** as seen earlier.

Alternative Derivations/Multiplication

Multiplication can be carried out using *any* alternate pair u,,v in the multiplying sequence, giving multipliers −u,v ; but only the root values are *non-negative* pairs. For example: let P=(3,1) and S=(2,1):

S=(2,1):	…	5	−2	[3]	1	4	5	…
P=(3,1):	−10	7	−3	[4]	1	5	6	11
(3,1)(−5,3):		41	−23	18	−5	[13]	8	
(3,1)(2,1):		**−23**	**18**	**−5**	**[13]**	**8**	**21**	
(3,1)(−3,4):		18	−5	[13]	8	21	29	
(3,1)(−1,5)		−5	[13]	8	21	29	50	

The same sequence results, but the terms shift diagonally in the table.

If we multiply sequence X by sequence Y, the general term in the resulting array can be expressed as Z[i][j] = −X(i−1)Y(j−1) + X(i+1)Y(j+1).
The diagonal shift can be expressed as Z[i+1][j] = Z[i][j+1], and the proof:

Z[i+1][j] = −X(i)Y(j−1) + X(i+2)Y(j+1) = −X(i)Y(j−1) + (X(i) + X(i+1))Y(j+1)
= X(i)(Y(j+1) – Y(j−1)) + X(i+1)Y(j+1) = **X(i)Y(j) + X(i+1))Y(j+1)**

Z[i][j+1] = −X(i−1)Y(j) + X(i+1)Y(j+2) = −X(i−1)Y(j) + X(i+1)(Y(j) + Y(j+1))
= (X(i+1) – X(i−1))Y(j) + X(i+1)Y(j+1) = **X(i)Y(j) + X(i+1))Y(j+1)**

yields another relationship: Z[i+1][j] = Z[i][j+1] = **X(i)Y(j) + X(i+1)Y(j+1)**
which expresses multiplication as *consecutive* terms, such as starting values, multiplying *consecutive* terms, which may be easier to visualise in the table.

Example of Sequence Products

Take the sequences P=(7,5) and S=(3,2) which, with their complements and order of multiplication, generate eight possible products:

Fibonacci	5	−3	2	−1	1	[0]	1	1	2	3	5	$K_{1,0}$=1
P=(7,5)		45	−26	19	−7	[12]	5	17	22	39		K=179
S=(3,2)		19	−11	8	−3	[5]	2	7	9	16		K=31
(7,5)(3,2)			173	−92	81	−11	[70]	59	129			5549
(7,5)(2,3)			147	−73	[74]	1	75	76	151			5549
(5,7)(3,2)			151	−76	75	−1	[74]	73	147			5549
(5,7)(2,3)			129	−59	[70]	11	81	92	173			5549
(3,2)(7,5)			173	−92	81	−11	[70]	59	129			5549
(3,2)(5,7)			151	−76	75	−1	[74]	73	147			5549
(2,3)(7,5)			147	−73	[74]	1	75	76	151			5549
(2,3)(5,7)			129	−59	[70]	11	81	92	173			5549

Several things become apparent from the table above:

(i) The Cassini constants of the products are all the same, being in fact the product of the Cassini constants: 5549 = 179 × 31.

(ii) Sequence multiplication is commutative, e.g. (7,5)(3,2) = (3,2)(7,5).

(iii) The product of the complements of two sequences is the complement of the product of the sequences, e.g. (7,5)(3,2) = (11,59) and (2,3)(5,7) = (59,11).

(iv) The index shifts one place left or right when multiplying two non-complementary sequences.

(v) The 8 results contain just two unique Primary sequences: (59,11) and (73,1).

Properties of Sequence Products

We now prove that **sequence multiplication is commutative**, and that the **Cassini constant of the product** of two sequences is the **product of the Cassini constants** of the two sequences.

Repeating the earlier multiplication table for PS=(p,q)×(r,s):

(p,q)(r,s):	...	2pr+qr+ps+qs	−pr+qs	pr+qr+ps+2qs	...

If we swap both p⟷r and q⟷s then the terms pr and qs are unchanged; and the terms ps and qr are paired, so sequence multiplication is commutative.

The three derived terms give us the Cassini constant of the product:

$\mathbf{K_{PS}} = (2pr + qr + ps + qs)(pr + qr + ps + 2qs) - (-pr + qs)^2$
$= 2p^2r^2 + 2pqr^2 + 2p^2rs + 4pqrs + pqr^2 + q^2r^2 + pqrs + 2q^2rs$
$\quad + p^2rs + pqrs + p^2s^2 + 2pqs^2 + pqrs + q^2rs + pqs^2 + 2q^2s^2$
$\quad - p^2r^2 + 2pqrs - q^2s^2$
$= p^2r^2 + 3p^2rs + p^2s^2 + 3pqr^2 + 9pqrs + 3pqs^2 + q^2r^2 + 3q^2rs + q^2s^2$
$= (p^2 + 3pq + q^2)(r^2 + 3rs + s^2) = \mathbf{K_P K_S}$ [QED]

The result $\mathbf{K_P K_S = K_{PS}}$ leads immediately to the following observation:

OEIS® A031363 is Closed Under Multiplication

An integer in A031363 is of the form $a(a+b) - b^2$ and so is the Cassini constant of ${}^aS_b$, with root values r≥0 and s≥0 say, but not both zero.

Therefore two integers I,J in A031363 each relate to at least one Fibonacci-like sequence, say P=(p,q) where K_P=I and S=(r,s) where K_S=J. The product of these sequences is another sequence PS=(x,y) whose Cassini constant K_{PS} is necessarily in A031363. But $K_{PS} = K_P K_S$ = IJ, so IJ is in A031363. [QED]

We have shown that for any integers p,q,r,s ≥ 0 there are integers x,y ≥ 0 such that $(x^2 + 3xy + y^2)=(p^2 + 3pq + q^2)(r^2 + 3rs + s^2)$.
Next we work out how to find x,y in terms of p,q,r,s.

Root Values and Index Origin of a Sequence Product

The table for the multiplication of P=(p,q) by S=(r,s) is:

(p,q):	−(3p+q)	2p+q	−p	[p+q]	q	p+2q	p+3q
(p,q)(r,s):	…	−(3pr+qr+ps)	(N/A)	−pr+qs	(N/A)	qr+ps+3qs	…

Because each product term derives from an alternate pair, the results to the right are always positive and the results to the left alternate positive and negative. The critical term is (−pr + qs), i.e. the one "under" the origin of (p,q) because it is the only one derived from terms of opposite signs (root terms).

If (−pr + qs) = 0 then pr = qs.
We need the following result from number theory, where "a|b" is notation for "a divides b": *"if x|yz but x and y are relatively prime then x|z".*
Let (p,q)=λ(p',q') where λ is the hcf of p and q; and let (r,s)=μ(r',s') where μ is the hcf of r and s. Then p'r'=q's', so p'|q's', but p' and q' are relatively prime so p'|s'; similarly s'|p'r' but r' and s' are relatively prime so s'|p'.
Hence p'=s' and q'=r', so (p',q')=(s',r') =T say. Therefore P=λT and S=μT', so:
P and S are scalar multiples of complementary sequences, and the product **PS=(λμK_T)F is a scalar multiple of F** – with origin [0] unshifted in the table.

But **generally, multiplication causes the origin to shift one place in the table**:

If (−pr + qs) < 0 it is the *left* of the pair with opposite signs so has index "−1", **the root values are (pr − qs),(qr + ps + 3qs)**. The origin shifts one place *right*.

If (−pr + qs) > 0 it is the *right* of the pair with opposite signs so has index "+1", **the root values are (3pr + qr + ps),(qs − pr)**. The origin shifts one place *left*.

In either non-zero case |pr − qs| is one of the root values. Say it is x, then y can be calculated from $x^2 + 3xy + y^2 = K$ (where $K=K_{PS}$), giving the quadratic:
$y^2 + 3xy + x^2 - K = 0$ and $y = (-3x \pm \surd(9x^2 - 4(x^2 - K))/2 = (-3x \pm \surd(5x^2 + 4K))/2$.
Note that $(5x^2 + 4K)$ must be a perfect square, and turns out to be $(3x + 2y)^2$.

I.e. the **root values of (p,q)(r,s)** are **x=|pr − qs|** and **$y = (-3x + \surd(5x^2 + 4K_{PS}))/2$**.

Using the examples of sequence multiplication in the earlier table: K=5549 and (7,5)(3,2) gives x=11 and y=59 while (7,5)(2,3) gives x=|−1|=1 and y=73.

Product of Two Primary Sequences

The product of two primary sequences may or may not be a primary sequence. If p>q and r>s then −pr+qs<0. For the product to be primary we need: pr−qs > qr+ps+3qs i.e. pr > qr+ps+4qs, which happens when *both* p is sufficiently greater than q *and* r is sufficiently greater than s.

For example: (3,1)(3,1)=(8,9) is secondary but (4,1)(3,1)=(11,10) is primary.

Further: (10,1)(3,2)=(28,29) is secondary and (12,1)(3,2)=(34,33) is primary, while the middle product (11,1)(3,2)=(31,31) is the scalar multiple 31(1,1). This is because (11,1) is itself the product (2,3)(1,1), so the combination gives a product of complementary sequences: (3,2)(2,3)(1,1) = 31(1,1) where 31=$_3K_2$.

We rearranged the terms above using commutativity and the following result:

Sequence Multiplication is Associative

We show that ((p,q)(r,s))(u,v) = (p,q)((r,s)(u,v)):

For ((p,q)(r,s))(u,v) we get:

(p,q):	−(3p+q)	2p+q	−p	[p+q]	q	p+2q	p+3q
(p,q)(r,s):	…	−(3pr+qr+ps)	(N/A)	−pr+qs	(N/A)	qr+ps+3qs	…
((p,q)(r,s))(u,v):	…	…	−(3pr+qr+ps)u + (−pr+qs)v	(N/A)	(−pr+qs)u + (qr+ps+3qs)v	…	…

For (p,q)((r,s)(u,v)) we commute to ((r,s)(u,v))(p,q):

(r,s):	−(3r+s)	2r+s	−r	[r+s]	s	r+2s	r+3s
(r,s)(u,v):	…	−(3ru+su+rv)	(N/A)	−ru+sv	(N/A)	su+rv+3sv	…
((r,s)(u,v))(p,q):	…	…	−(3ru+su+rv)p + (−ru+sv)q	(N/A)	(−ru+sv)p (su+rv+3sv)q	…	…

Being −(3pru + qru + psu + prv − qsv) and (−pru + qsu + qrv + psv + 3qsv), the two derived terms are the same for both products. They may not be the roots but they do define the sequences, so ((p,q)(r,s))(u,v) = (p,q)((r,s)(u,v)). [QED]

The Product of the Complements of Sequences

We can now prove what we saw from the examples in the earlier table:
The **product of the complements** of two sequences is the **complement of the product** of the sequences, or (PS)'=P'S': if (a,b)(c,d)=(x,y) then (b,a)(d,c)=(y,x), which commutes to (y,x)=(d,c)(b,a) and (PS)'=S'P'.

Suppose $(-ac + bd) < 0$. (A similar argument applies if $(-ac + bd) > 0$.)

For the complement of the product, let p=a, q=b, r=c, s=d.
Then $(-pr+qs)=(-ac+bd)<0$, giving roots $(pr - qs),(qr + ps + 3qs)$ which become $(ac - bd),(bc + ad + 3bd)$ and so the complement of the product has root values $(bc + ad + 3bd),(ac - bd)$.

For the product of the complements (b,a)(d,c), let p=b, q=a, r=d, s=c.
Then $(-pr+qs)=(-bd+ac)>0$, giving roots $(3pr + qr + ps),(qs - pr)$ which become $(3bd + ad + bc),(ac - bd)$. These are the same as above. [QED]

We can extend this to three or more sequences, i.e. (PSU)' = U'S'P' :
(PSU)' = ((PS)U)' = (PS)'U' = (P'S')U' = P'S'U' = U'S'P'.

Terminology/Classification of Sequences

Having looked at sequence multiplication we next turn to Factorisation, and will use the following terminology:

A **prime sequence** is one with a prime Cassini constant, i.e. one which is in A038872. Its starting and root values are necessarily relatively prime.

A **relatively prime sequence** has relatively prime starting and root values and its Cassini constant is in A089270. This may or may not be prime.
A prime sequence is also a relatively prime sequence.

We previously defined a **scalar multiple** sequence as one in which every term has a common factor λ. Its Cassini constant has the factor λ^2, and is in A031363 but not A038872 (the prime Cassini constants), and may or may not be in A089270. Scalar multiple sequences are mutually exclusive with the relatively prime sequences (including prime sequences).

Sequence Factorisation and Number Theory Conjectures

If a sequence S has sequence factors X and Y then $K_S=K_XK_Y$, so to factorise S we look at the numerical factors of K_S. Since different sequences can have the same Cassini constant, K_S alone is not enough – we need the root values of S.

If we know the root (or starting) values of a sequence we can pull out any common factor and express the sequence as λS, or λ(r,s) where r and s are relatively prime, and its Cassini constant is λ^2K_S.

A computer program generated Cassini constants for all starting values a=r+s from 1 to 10,000 and b≤a/2 (r≥s). The results included 3,317,555 prime numbers, the last being for (r,s)=(5003,4997) giving prime K=124,999,991. There were **no** duplicate primes, which suggests:

Conjecture #1

If $K=(r^2 + 3rs + s^2)$ is a prime > 5, then r>s>0 are uniquely determined.

In other words, **a prime Cassini constant** (a number in OEIS® A038872) **corresponds to just one pair of complementary sequences (r,s) and (s,r)**, or if K=5 to the Lucas sequence (r,s)=(1,1).

For ${}_rK_s$ to be prime r and s must be relatively prime, but it is neither necessary nor sufficient that either or both be prime; for example: ${}_3K_2=31$, ${}_4K_3=61$ and ${}_{15}K_4=421$ are all prime, while ${}_7K_2=95$, ${}_8K_5=209$ and ${}_9K_4=205$ are all composite.

The listing earlier and Appendix A show sequences generated from relatively prime roots, with their Cassini constants. Where a constant is not prime its prime factors are shown, and we observe that each of these factors appears earlier in the listing as Cassini constants of prime sequences. This suggests:

Conjecture #2a

If u and v are relatively prime then $K=(u^2 + 3uv + v^2)$ is either prime or is the product of prime factors each of the same form.

In other words, if u,v are relatively prime but K is not prime then it has factors $K=(p^2 + 3pq + q^2)(r^2 + 3rs + s^2)$ where p,q,r,s>0; and since the pairs p,q and r,s must each be relatively prime this process can be repeated until all the factors are prime and of the same form. Appendix B shows examples of this.

Since each prime factor corresponds to a prime sequence, we have:

Conjecture #2b

Every sequence is a scalar multiple (possibly 1) of the product of prime sequences. This factorisation is unique, apart from order.

The factor 5 corresponds only to the Lucas sequence (1,1). Other prime factors each correspond to one of two complementary prime sequences. Given the root values of a sequence S we can calculate K_S, find its prime factors and form the products of all combinations of the corresponding prime sequences.
One, and only one, of these will be the sequence S.

For example the sequence (12,17): … 41 −12 [29] 17 46 … has K=1045
= 5 × 11 × 19, where $5={}_1K_1(=K_L)$, $11={}_2K_1={}_1K_2$ and $19={}_3K_1={}_1K_3$, so can result from four different sequences, in two complementary pairs: (1,1)(2,1)(3,1)=(28,3) , (1,1)(2,1)(1,3)=(12,17) , (1,1)(1,2)(3,1)=(17,12) and (1,1)(1,2)(1,3)=(3,28) .
So the required prime sequence factorisation is: **(12,17)=(1,1)(2,1)(1,3)** .
(The non-prime factorisations are (12,17)=(1,1)(13,1)=(2,1)(7,2)=(1,3)(1,6).)

The factor 5 can appear only once, since $L^2=5F$ would make a scalar multiple.
Other prime factors can appear to any power, which we now consider:

Sequences with Cassini Constant $({}_rK_s)^n$, where r≠s are Relatively Prime

$({}_rK_s)^n=({}_sK_r)^n$ is the Cassini constant of sequences with n factors, each either (r,s) or (s,r). Of the 2^n combinations there are n+1 different products of the form $(r,s)^{n-c}(s,r)^c$ where c=0,...,n. If c is neither 0 nor n, the product contains complementary pairs and is a scalar multiple: if c<n/2 it is $({}_rK_s)^c(r,s)^{n-2c}$,
if c>n/2 it is $({}_rK_s)^{n-c}(s,r)^{2c-n}$, and if n is even and c=n/2 it is $({}_rK_s)^cF$.

Hence only $(r,s)^n$ and its complement $(s,r)^n$ can be relatively prime sequences.

The following are examples for $({}_3K_1)^2=19^2$ and $({}_2K_1)^3=11^3$.

$({}_3K_1)^2$:	$(3,1)^2$ = (8,9)	19F	$(1,3)^3$ = (9,8)	
$({}_2K_1)^3$:	$(2,1)^3$ = (35,1)	11(2,1)	11(1,2)	$(1,2)^3$ = (1,35)

Sequences Which Have A Given Cassini Constant

We can now determine the number and form of all the sequences which have a given Cassini Constant K, and hence the solutions to $(r^2 + 3rs + s^2) = K$.

A general Cassini constant, like any number, can be factorised into a product of powers of primes: Π (p_i^n_i), where "^" means "to the power of".

We start with F as the identity or null-product sequence for the first factor, with scalar multiplier $\lambda=1$ and number of sequences N=1.
With p_i in ascending order we process each term p_i^n_i as follows:

1) If p_i is not a prime Cassini constant (i.e. not in A038872), then it can only arise from a scalar multiple and n_i must be even, say $n_i=2c_i$. We multiply λ by p_i^c_i , but the sequence factors and number of sequences N remain the same.

2) If p_i is 5, it corresponds to L=(1,1) which can appear as a sequence factor at most once; but 5 can be a numerical factor to any power.
If n_i is odd, replace the first sequence factor F by L=(1,1).
If $n_i > 1$, the other factors 5 can only arise from scalar multiples ($L^2=5F$).
Express n_i as $2c_i$ or ($2c_i$ +1) as appropriate, and multiply λ by 5^c_i.
In all cases, the number of sequences N remains the same.

3) If p_i is a prime Cassini constant other than 5, then there are (n_i +1) possible choices for the next sequence factor, as discussed above. Multiply N by (n_i +1).
If $n_i > 1$, all but 2 of these choices result in a further scalar multiple factor, which applies only to sequence products with that choice.

Each of the N resulting sequences can be evaluated to give a unique (r,s), and for each (r,s) there will also be a corresponding complement (s,r).

Examples

Above we looked at the cases for K=1045=5×11×19 where $\lambda=1$ and N=4:
(1,1)(2,1)(3,1) , (1,1)(2,1)(1,3) , (1,1)(1,2)(3,1) and (1,1)(1,2)(1,3) ,
giving sequences: (28,3) , (12,17) , (17,12) and (3,28).

For K=5225=5^2× 11×19, these become $\lambda=5$ and N=4:
5(2,1)(3,1)=(25,40) , 5(2,1)(1,3)=(65,5) , 5(1,2)(3,1)=(5,65) , 5(1,2)(1,3)=(40,25).

For K=480491=11^3×19^2 we get N=12 and the following possible sequences shown in table form, where the relatively prime results are highlighted:

$(2,1)^3(3,1)^2$ = **(271,350)**	$11(2,1)(3,1)^2$ = (77,583)	$11(1,2)(3,1)^2$ = (539,110)	$(1,2)^3(3,1)^2$ = **(313,307)**
$19(2,1)^3$ = (665,19)	11×19(2,1) = (418,209)	11×19(1,2) = (209,418)	$19(1,2)^3$ = (19,665)
$(2,1)^3(1,3)^2$ = **(307,313)**	$11(2,1)(1,3)^2$ = (110,539)	$11(1,2)(1,3)^2$ = (583,77)	$(1,2)^3(1,3)^2$ = **(350,271)**

So the 12 solutions to $r^2 + 3rs + s^2 = 480491$ are the r,s and s,r pairs:,
 665 19 , 583 77 , 539 110 , 418 209 , **350 271** and **313 307** .

The number of relatively prime solutions to $(r^2 + 3rs + s^2) = K$ depends only on the number of distinct prime factors other than 5 and **not** on their powers. Each such prime (a number in A045468) doubles the possibilities.

Sequences $(r,s)^n$, where r≠s are Relatively Prime

The following tables show some powers of (2,1), with root terms highlighted:

${}_2S_1$=(2,1)	−7	5	**−2**	[3]	**1**	4	5	9	14	K=11	Primary
(2,1)(2,1)		−16	13	**−3**	[10]	**7**	17	24		K=121	Secondary
$(2,1)^3$			**−35**	[36]	**1**	37	38			K=1331	Primary
$(2,1)^4$				**−69**	[109]	**40**				$K=11^4$	Primary

Continuing:

$(2,1)^4$				**−69**	[109]	**40**			$K=11^4$	Primary
$(2,1)^5$					**−98**	[367]	**269**		$K=11^5$	Secondary
$(2,1)^6$				**−1224**	[1297]	**73**			$K=11^6$	Primary
$(2,1)^7$					**−2375**	[3964]	**1589**		$K=11^7$	Primary
$(2,1)^8$						**−3161**	[13481]	**10320**	$K=11^8$	Secondary
$(2,1)^9$					**−42767**	[46765]	**3998**		$K=11^9$	Primary

Note that all the resulting sequences are relatively prime.

For $(r,s)^n$ to be a relatively prime sequence it is necessary for r and s to be relatively prime. This example suggests that this is also a sufficient condition:

Conjecture #3a

If (r,s) is a relatively prime sequence other than (1,1), (1,0) or (0,1), then $(r,s)^n$ is a relatively prime sequence.

In other words: apart from the Fibonacci sequence, which acts as an identity, and the Lucas sequence which is its own complement, every power of a relatively prime sequence is also a (different) relatively prime sequence.

Appendix C shows examples of products of pairs of relatively prime sequences. The only products which are scalar multiples are products of complementary pairs, or of sequences which appear as products earlier in the list and between them have a complementary pair of prime factors. This suggests:

Conjecture #3b

If (p,q) and (r,s) are both relatively prime sequences, then the product sequence (p,q)(r,s) is relatively prime if and only if (p,q) and (r,s) between them have no complementary pair of prime sequences as factors.

Corollary

The product of any two numbers in A089270 which are not both divisible by 5 is also in A089270.

The proof is an extension of the proof of the closure of A031363.
Two integers I,J in A089270 each relate to at least one relatively prime sequence, say P where K_P=I and S where K_S=J. Each of these sequences can be factorised into a product of prime sequences. If S has any prime factor (v,u) which is the complement of a prime factor (u,v) of P it cannot be (1,1), or the Cassini constants of both sequences would be divisible by 5. Since u≠v we can create a new sequence T which has factor (u,v) instead of (v,u), and K_T=K_S unchanged. Repeat this process until T has no complement of any factor of P. The product of P and T is relatively prime by the conjecture, and so K_{PT} is in A089270. But $K_{PT} = K_P K_T = K_P K_S$ = IJ, so IJ is in A089270. [QED]

Sequence A336403 and its Closure Under Multiplication

A336403 is the subsequence of A089270 which excludes terms divisible by 5.

The proof above applies to show **A336403 is closed under multiplication**.

A336403 can also be defined as: “Numbers $r^2 + 3rs + s^2$ not divisible by 5, with r and s relatively prime”, and can be further limited: “with r > s ≥ 0”.

Properties of Related Sequences

A031363 consists of all, and only, Cassini constants. It has a natural one-to-one mapping with the Cartesian product of A000290 (the squares) and A089270.	
A089270 is the subsequence of above with relatively prime roots.	**A336403** is A089270 without 5 and all multiples of 5.
A038872 is the prime number subsequence of above.	**A045468** is the prime number subsequence of above, and is A033872 without the 5.

The numbers in A089270 all have prime factors in A038872, and the product of any numbers in A038872, except 5, each to any power is in A089270. The prime 5 may also appear, but just once, in any such product.

The numbers in A336403 all have prime factors in A045468, and the product of any numbers in A045468, each to any power, is in A336403.

Appendix A

Examples of Fibonacci-Like Sequences, Their Cassini Constants and Factors

This list is an extension of the examples shown earlier in the document, and has starting values in the range [20] through [55].
Entries are listed the order of their starting values **[a],b** with **a** the major number, with **a** and **b** relatively prime. We restrict **b** to not more than **a/2.**
We show a few terms either side of the origin to start the sequence and its complement. We also show the Cassini constant K and its factors.
Note that K is in ascending order *only* within each major starting value [a].
For each value of a, K has bounds: $a^2 < K < (5a^2)/4$.

... 33 -14 [19] 5 24 ... K=431 (prime)
... 32 -13 [19] 6 25 ... K=439 (prime)
... 31 -12 [19] 7 26 ... K=445 = 5 x 89
... 30 -11 [19] 8 27 ... K=449 (prime)
... 29 -10 [19] 9 28 ... K=451 = 11 x 41
... 39 -19 [20] 1 21 ... K=419 (prime)
... 37 -17 [20] 3 23 ... K=451 = 11 x 41
... 33 -13 [20] 7 27 ... K=491 (prime)
... 31 -11 [20] 9 29 ... K=499 (prime)
... 41 -20 [21] 1 22 ... K=461 (prime)
... 40 -19 [21] 2 23 ... K=479 (prime)
... 38 -17 [21] 4 25 ... K=509 (prime)
... 37 -16 [21] 5 26 ... K=521 (prime)
... 34 -13 [21] 8 29 ... K=545 = 5 x 109
... 32 -11 [21] 10 31 ... K=551 = 19 x 29
... 43 -21 [22] 1 23 ... K=505 = 5 x 101
... 41 -19 [22] 3 25 ... K=541 (prime)
... 39 -17 [22] 5 27 ... K=569 (prime)
... 37 -15 [22] 7 29 ... K=589 = 19 x 31
... 35 -13 [22] 9 31 ... K=601 (prime)
... 45 -22 [23] 1 24 ... K=551 = 19 x 29
... 44 -21 [23] 2 25 ... K=571 (prime)
... 43 -20 [23] 3 26 ... K=589 = 19 x 31
... 42 -19 [23] 4 27 ... K=605 = 5 x 11 x 11
... 41 -18 [23] 5 28 ... K=619 (prime)
... 40 -17 [23] 6 29 ... K=631 (prime)
... 39 -16 [23] 7 30 ... K=641 (prime)
... 38 -15 [23] 8 31 ... K=649 = 11 x 59
... 37 -14 [23] 9 32 ... K=655 = 5 x 131
... 36 -13 [23] 10 33 ... K=659 (prime)
... 35 -12 [23] 11 34 ... K=661 (prime)
... 47 -23 [24] 1 25 ... K=599 (prime)
... 43 -19 [24] 5 29 ... K=671 = 11 x 61
... 41 -17 [24] 7 31 ... K=695 = 5 x 139
... 37 -13 [24] 11 35 ... K=719 (prime)
... 49 -24 [25] 1 26 ... K=649 = 11 x 59
... 48 -23 [25] 2 27 ... K=671 = 11 x 61
... 47 -22 [25] 3 28 ... K=691 (prime)
... 46 -21 [25] 4 29 ... K=709 (prime)
... 44 -19 [25] 6 31 ... K=739 (prime)
... 43 -18 [25] 7 32 ... K=751 (prime)
... 42 -17 [25] 8 33 ... K=761 (prime)
... 41 -16 [25] 9 34 ... K=769 (prime)
... 39 -14 [25] 11 36 ... K=779 = 19 x 41
... 38 -13 [25] 12 37 ... K=781 = 11 x 71
... 51 -25 [26] 1 27 ... K=701 (prime)
... 49 -23 [26] 3 29 ... K=745 = 5 x 149
... 47 -21 [26] 5 31 ... K=781 = 11 x 71
... 45 -19 [26] 7 33 ... K=809 (prime)
... 43 -17 [26] 9 35 ... K=829 (prime)
... 41 -15 [26] 11 37 ... K=841 = 29 x 29
... 53 -26 [27] 1 28 ... K=755 = 5 x 151
... 52 -25 [27] 2 29 ... K=779 = 19 x 41
... 50 -23 [27] 4 31 ... K=821 (prime)
... 49 -22 [27] 5 32 ... K=839 (prime)
... 47 -20 [27] 7 34 ... K=869 = 11 x 79
... 46 -19 [27] 8 35 ... K=881 (prime)
... 44 -17 [27] 10 37 ... K=899 = 29 x 31
... 43 -16 [27] 11 38 ... K=905 = 5 x 181
... 41 -14 [27] 13 40 ... K=911 (prime)
... 55 -27 [28] 1 29 ... K=811 (prime)
... 53 -25 [28] 3 31 ... K=859 (prime)
... 51 -23 [28] 5 33 ... K=899 = 29 x 31
... 47 -19 [28] 9 37 ... K=955 = 5 x 191
... 45 -17 [28] 11 39 ... K=971 (prime)
... 43 -15 [28] 13 41 ... K=979 = 11 x 89
... 57 -28 [29] 1 30 ... K=869 = 11 x 79
... 56 -27 [29] 2 31 ... K=895 = 5 x 179
... 55 -26 [29] 3 32 ... K=919 (prime)
... 54 -25 [29] 4 33 ... K=941 (prime)
... 53 -24 [29] 5 34 ... K=961 = 31 x 31
... 52 -23 [29] 6 35 ... K=979 = 11 x 89
... 51 -22 [29] 7 36 ... K=995 = 5 x 199
... 50 -21 [29] 8 37 ... K=1009 (prime)
... 49 -20 [29] 9 38 ... K=1021 (prime)
... 48 -19 [29] 10 39 ... K=1031 (prime)
... 47 -18 [29] 11 40 ... K=1039 (prime)
... 46 -17 [29] 12 41 ... K=1045 = 5 x 11 x 19
... 45 -16 [29] 13 42 ... K=1049 (prime)
... 44 -15 [29] 14 43 ... K=1051 (prime)

... 59 -29 [30] 1 31 ... K=929 (prime)
... 53 -23 [30] 7 37 ... K=1061 (prime)
... 49 -19 [30] 11 41 ... K=1109 (prime)
... 47 -17 [30] 13 43 ... K=1121 = 19 x 59
... 61 -30 [31] 1 32 ... K=991 (prime)
... 60 -29 [31] 2 33 ... K=1019 (prime)
... 59 -28 [31] 3 34 ... K=1045 = 5 x 11 x 19
... 58 -27 [31] 4 35 ... K=1069 (prime)
... 57 -26 [31] 5 36 ... K=1091 (prime)
... 56 -25 [31] 6 37 ... K=1111 = 11 x 101
... 55 -24 [31] 7 38 ... K=1129 (prime)
... 54 -23 [31] 8 39 ... K=1145 = 5 x 229
... 53 -22 [31] 9 40 ... K=1159 = 19 x 61
... 52 -21 [31] 10 41 ... K=1171 (prime)
... 51 -20 [31] 11 42 ... K=1181 (prime)
... 50 -19 [31] 12 43 ... K=1189 = 29 x 41
... 49 -18 [31] 13 44 ... K=1195 = 5 x 239
... 48 -17 [31] 14 45 ... K=1199 = 11 x 109
... 47 -16 [31] 15 46 ... K=1201 (prime)
... 63 -31 [32] 1 33 ... K=1055 = 5 x 211
... 61 -29 [32] 3 35 ... K=1111 = 11 x 101
... 59 -27 [32] 5 37 ... K=1159 = 19 x 61
... 57 -25 [32] 7 39 ... K=1199 = 11 x 109
... 55 -23 [32] 9 41 ... K=1231 (prime)
... 53 -21 [32] 11 43 ... K=1255 = 5 x 251
... 51 -19 [32] 13 45 ... K=1271 = 31 x 41
... 49 -17 [32] 15 47 ... K=1279 (prime)
... 65 -32 [33] 1 34 ... K=1121 = 19 x 59
... 64 -31 [33] 2 35 ... K=1151 (prime)
... 62 -29 [33] 4 37 ... K=1205 = 5 x 241
... 61 -28 [33] 5 38 ... K=1229 (prime)
... 59 -26 [33] 7 40 ... K=1271 = 31 x 41
... 58 -25 [33] 8 41 ... K=1289 (prime)
... 56 -23 [33] 10 43 ... K=1319 (prime)
... 53 -20 [33] 13 46 ... K=1349 = 19 x 71
... 52 -19 [33] 14 47 ... K=1355 = 5 x 271
... 50 -17 [33] 16 49 ... K=1361 (prime)
... 67 -33 [34] 1 35 ... K=1189 = 29 x 41
... 65 -31 [34] 3 37 ... K=1249 (prime)
... 63 -29 [34] 5 39 ... K=1301 (prime)
... 61 -27 [34] 7 41 ... K=1345 = 5 x 269
... 59 -25 [34] 9 43 ... K=1381 (prime)
... 57 -23 [34] 11 45 ... K=1409 (prime)
... 55 -21 [34] 13 47 ... K=1429 (prime)
... 53 -19 [34] 15 49 ... K=1441 = 11 x 131
... 69 -34 [35] 1 36 ... K=1259 (prime)
... 68 -33 [35] 2 37 ... K=1291 (prime)
... 67 -32 [35] 3 38 ... K=1321 (prime)
... 66 -31 [35] 4 39 ... K=1349 = 19 x 71
... 64 -29 [35] 6 41 ... K=1399 (prime)
... 62 -27 [35] 8 43 ... K=1441 = 11 x 131
... 61 -26 [35] 9 44 ... K=1459 (prime)
... 59 -24 [35] 11 46 ... K=1489 (prime)
... 58 -23 [35] 12 47 ... K=1501 = 19 x 79
... 57 -22 [35] 13 48 ... K=1511 (prime)
... 54 -19 [35] 16 51 ... K=1529 = 11 x 139
... 53 -18 [35] 17 52 ... K=1531 (prime)
... 71 -35 [36] 1 37 ... K=1331 = 11 x 11 x 11
... 67 -31 [36] 5 41 ... K=1451 (prime)
... 65 -29 [36] 7 43 ... K=1499 (prime)
... 61 -25 [36] 11 47 ... K=1571 (prime)
... 59 -23 [36] 13 49 ... K=1595 = 5 x 11 x 29
... 55 -19 [36] 17 53 ... K=1619 (prime)
... 73 -36 [37] 1 38 ... K=1405 = 5 x 281
... 72 -35 [37] 2 39 ... K=1439 (prime)
... 71 -34 [37] 3 40 ... K=1471 (prime)
... 70 -33 [37] 4 41 ... K=1501 = 19 x 79
... 69 -32 [37] 5 42 ... K=1529 = 11 x 139
... 68 -31 [37] 6 43 ... K=1555 = 5 x 311
... 67 -30 [37] 7 44 ... K=1579 (prime)
... 66 -29 [37] 8 45 ... K=1601 (prime)
... 65 -28 [37] 9 46 ... K=1621 (prime)
... 64 -27 [37] 10 47 ... K=1639 = 11 x 149
... 63 -26 [37] 11 48 ... K=1655 = 5 x 331
... 62 -25 [37] 12 49 ... K=1669 (prime)
... 61 -24 [37] 13 50 ... K=1681 = 41 x 41
... 60 -23 [37] 14 51 ... K=1691 = 19 x 89
... 59 -22 [37] 15 52 ... K=1699 (prime)
... 58 -21 [37] 16 53 ... K=1705 = 5 x 11 x 31
... 57 -20 [37] 17 54 ... K=1709 (prime)
... 56 -19 [37] 18 55 ... K=1711 = 29 x 59
... 75 -37 [38] 1 39 ... K=1481 (prime)
... 73 -35 [38] 3 41 ... K=1549 (prime)
... 71 -33 [38] 5 43 ... K=1609 (prime)
... 69 -31 [38] 7 45 ... K=1661 = 11 x 151
... 67 -29 [38] 9 47 ... K=1705 = 5 x 11 x 31
... 65 -27 [38] 11 49 ... K=1741 (prime)
... 63 -25 [38] 13 51 ... K=1769 = 29 x 61
... 61 -23 [38] 15 53 ... K=1789 (prime)
... 59 -21 [38] 17 55 ... K=1801 (prime)
... 77 -38 [39] 1 40 ... K=1559 (prime)
... 76 -37 [39] 2 41 ... K=1595 = 5 x 11 x 29
... 74 -35 [39] 4 43 ... K=1661 = 11 x 151
... 73 -34 [39] 5 44 ... K=1691 = 19 x 89
... 71 -32 [39] 7 46 ... K=1745 = 5 x 349
... 70 -31 [39] 8 47 ... K=1769 = 29 x 61
... 68 -29 [39] 10 49 ... K=1811 (prime)
... 67 -28 [39] 11 50 ... K=1829 = 31 x 59
... 64 -25 [39] 14 53 ... K=1871 (prime)
... 62 -23 [39] 16 55 ... K=1889 (prime)
... 61 -22 [39] 17 56 ... K=1895 = 5 x 379
... 59 -20 [39] 19 58 ... K=1901 (prime)
... 79 -39 [40] 1 41 ... K=1639 = 11 x 149
... 77 -37 [40] 3 43 ... K=1711 = 29 x 59
... 73 -33 [40] 7 47 ... K=1831 (prime)
... 71 -31 [40] 9 49 ... K=1879 (prime)
... 69 -29 [40] 11 51 ... K=1919 = 19 x 101
... 67 -27 [40] 13 53 ... K=1951 (prime)
... 63 -23 [40] 17 57 ... K=1991 = 11 x 181
... 61 -21 [40] 19 59 ... K=1999 (prime)
... 81 -40 [41] 1 42 ... K=1721 (prime)
... 80 -39 [41] 2 43 ... K=1759 (prime)
... 79 -38 [41] 3 44 ... K=1795 = 5 x 359
... 78 -37 [41] 4 45 ... K=1829 = 31 x 59

... 77 -36 [41] 5 46 ... K=1861 (prime)
... 76 -35 [41] 6 47 ... K=1891 = 31 x 61
... 75 -34 [41] 7 48 ... K=1919 = 19 x 101
... 74 -33 [41] 8 49 ... K=1945 = 5 x 389
... 73 -32 [41] 9 50 ... K=1969 = 11 x 179
... 72 -31 [41] 10 51 ... K=1991 = 11 x 181
... 71 -30 [41] 11 52 ... K=2011 (prime)
... 70 -29 [41] 12 53 ... K=2029 (prime)
... 69 -28 [41] 13 54 ... K=2045 = 5 x 409
... 68 -27 [41] 14 55 ... K=2059 = 29 x 71
... 67 -26 [41] 15 56 ... K=2071 = 19 x 109
... 66 -25 [41] 16 57 ... K=2081 (prime)
... 65 -24 [41] 17 58 ... K=2089 (prime)
... 64 -23 [41] 18 59 ... K=2095 = 5 x 419
... 63 -22 [41] 19 60 ... K=2099 (prime)
... 62 -21 [41] 20 61 ... K=2101 = 11 x 191
... 83 -41 [42] 1 43 ... K=1805 = 5 x 19 x 19
... 79 -37 [42] 5 47 ... K=1949 (prime)
... 73 -31 [42] 11 53 ... K=2105 = 5 x 421
... 71 -29 [42] 13 55 ... K=2141 (prime)
... 67 -25 [42] 17 59 ... K=2189 = 11 x 199
... 65 -23 [42] 19 61 ... K=2201 = 31 x 71
... 85 -42 [43] 1 44 ... K=1891 = 31 x 61
... 84 -41 [43] 2 45 ... K=1931 (prime)
... 83 -40 [43] 3 46 ... K=1969 = 11 x 179
... 82 -39 [43] 4 47 ... K=2005 = 5 x 401
... 81 -38 [43] 5 48 ... K=2039 (prime)
... 80 -37 [43] 6 49 ... K=2071 = 19 x 109
... 79 -36 [43] 7 50 ... K=2101 = 11 x 191
... 78 -35 [43] 8 51 ... K=2129 (prime)
... 77 -34 [43] 9 52 ... K=2155 = 5 x 431
... 76 -33 [43] 10 53 ... K=2179 (prime)
... 75 -32 [43] 11 54 ... K=2201 = 31 x 71
... 74 -31 [43] 12 55 ... K=2221 (prime)
... 73 -30 [43] 13 56 ... K=2239 (prime)
... 72 -29 [43] 14 57 ... K=2255 = 5 x 11 x 41
... 71 -28 [43] 15 58 ... K=2269 (prime)
... 70 -27 [43] 16 59 ... K=2281 (prime)
... 69 -26 [43] 17 60 ... K=2291 = 29 x 79
... 68 -25 [43] 18 61 ... K=2299 = 11 x 11 x 19
... 67 -24 [43] 19 62 ... K=2305 = 5 x 461
... 66 -23 [43] 20 63 ... K=2309 (prime)
... 65 -22 [43] 21 64 ... K=2311 (prime)
... 87 -43 [44] 1 45 ... K=1979 (prime)
... 85 -41 [44] 3 47 ... K=2059 = 29 x 71
... 83 -39 [44] 5 49 ... K=2131 (prime)
... 81 -37 [44] 7 51 ... K=2195 = 5 x 439
... 79 -35 [44] 9 53 ... K=2251 (prime)
... 75 -31 [44] 13 57 ... K=2339 (prime)
... 73 -29 [44] 15 59 ... K=2371 (prime)
... 71 -27 [44] 17 61 ... K=2395 = 5 x 479
... 69 -25 [44] 19 63 ... K=2411 (prime)
... 67 -23 [44] 21 65 ... K=2419 = 41 x 59
... 89 -44 [45] 1 46 ... K=2069 (prime)
... 88 -43 [45] 2 47 ... K=2111 (prime)
... 86 -41 [45] 4 49 ... K=2189 = 11 x 199
... 83 -38 [45] 7 52 ... K=2291 = 29 x 79
... 82 -37 [45] 8 53 ... K=2321 = 11 x 211
... 79 -34 [45] 11 56 ... K=2399 (prime)
... 77 -32 [45] 13 58 ... K=2441 (prime)
... 76 -31 [45] 14 59 ... K=2459 (prime)
... 74 -29 [45] 16 61 ... K=2489 = 19 x 131
... 73 -28 [45] 17 62 ... K=2501 = 41 x 61
... 71 -26 [45] 19 64 ... K=2519 = 11 x 229
... 68 -23 [45] 22 67 ... K=2531 (prime)
... 91 -45 [46] 1 47 ... K=2161 (prime)
... 89 -43 [46] 3 49 ... K=2245 = 5 x 449
... 87 -41 [46] 5 51 ... K=2321 = 11 x 211
... 85 -39 [46] 7 53 ... K=2389 (prime)
... 83 -37 [46] 9 55 ... K=2449 = 31 x 79
... 81 -35 [46] 11 57 ... K=2501 = 41 x 61
... 79 -33 [46] 13 59 ... K=2545 = 5 x 509
... 77 -31 [46] 15 61 ... K=2581 = 29 x 89
... 75 -29 [46] 17 63 ... K=2609 (prime)
... 73 -27 [46] 19 65 ... K=2629 = 11 x 239
... 71 -25 [46] 21 67 ... K=2641 = 19 x 139
... 93 -46 [47] 1 48 ... K=2255 = 5 x 11 x 41
... 92 -45 [47] 2 49 ... K=2299 = 11 x 11 x 19
... 91 -44 [47] 3 50 ... K=2341 (prime)
... 90 -43 [47] 4 51 ... K=2381 (prime)
... 89 -42 [47] 5 52 ... K=2419 = 41 x 59
... 88 -41 [47] 6 53 ... K=2455 = 5 x 491
... 87 -40 [47] 7 54 ... K=2489 = 19 x 131
... 86 -39 [47] 8 55 ... K=2521 (prime)
... 85 -38 [47] 9 56 ... K=2551 (prime)
... 84 -37 [47] 10 57 ... K=2579 (prime)
... 83 -36 [47] 11 58 ... K=2605 = 5 x 521
... 82 -35 [47] 12 59 ... K=2629 = 11 x 239
... 81 -34 [47] 13 60 ... K=2651 = 11 x 241
... 80 -33 [47] 14 61 ... K=2671 (prime)
... 79 -32 [47] 15 62 ... K=2689 (prime)
... 78 -31 [47] 16 63 ... K=2705 = 5 x 541
... 77 -30 [47] 17 64 ... K=2719 (prime)
... 76 -29 [47] 18 65 ... K=2731 (prime)
... 75 -28 [47] 19 66 ... K=2741 (prime)
... 74 -27 [47] 20 67 ... K=2749 (prime)
... 73 -26 [47] 21 68 ... K=2755 = 5 x 19 x 29
... 72 -25 [47] 22 69 ... K=2759 = 31 x 89
... 71 -24 [47] 23 70 ... K=2761 = 11 x 251
... 95 -47 [48] 1 49 ... K=2351 (prime)
... 91 -43 [48] 5 53 ... K=2519 = 11 x 229
... 89 -41 [48] 7 55 ... K=2591 (prime)
... 85 -37 [48] 11 59 ... K=2711 (prime)
... 83 -35 [48] 13 61 ... K=2759 = 31 x 89
... 79 -31 [48] 17 65 ... K=2831 = 19 x 149
... 77 -29 [48] 19 67 ... K=2855 = 5 x 571
... 73 -25 [48] 23 71 ... K=2879 (prime)
... 97 -48 [49] 1 50 ... K=2449 = 31 x 79
... 96 -47 [49] 2 51 ... K=2495 = 5 x 499
... 95 -46 [49] 3 52 ... K=2539 (prime)
... 94 -45 [49] 4 53 ... K=2581 = 29 x 89
... 93 -44 [49] 5 54 ... K=2621 (prime)
... 92 -43 [49] 6 55 ... K=2659 (prime)
... 90 -41 [49] 8 57 ... K=2729 (prime)

... 89 -40 [49] 9 58 ... K=2761 = 11 x 251
... 88 -39 [49] 10 59 ... K=2791 (prime)
... 87 -38 [49] 11 60 ... K=2819 (prime)
... 86 -37 [49] 12 61 ... K=2845 = 5 x 569
... 85 -36 [49] 13 62 ... K=2869 = 19 x 151
... 83 -34 [49] 15 64 ... K=2911 = 41 x 71
... 82 -33 [49] 16 65 ... K=2929 = 29 x 101
... 81 -32 [49] 17 66 ... K=2945 = 5 x 19 x 31
... 80 -31 [49] 18 67 ... K=2959 = 11 x 269
... 79 -30 [49] 19 68 ... K=2971 (prime)
... 78 -29 [49] 20 69 ... K=2981 = 11 x 271
... 76 -27 [49] 22 71 ... K=2995 = 5 x 599
... 75 -26 [49] 23 72 ... K=2999 (prime)
... 74 -25 [49] 24 73 ... K=3001 (prime)
... 99 -49 [50] 1 51 ... K=2549 (prime)
... 97 -47 [50] 3 53 ... K=2641 = 19 x 139
... 93 -43 [50] 7 57 ... K=2801 (prime)
... 91 -41 [50] 9 59 ... K=2869 = 19 x 151
... 89 -39 [50] 11 61 ... K=2929 = 29 x 101
... 87 -37 [50] 13 63 ... K=2981 = 11 x 271
... 83 -33 [50] 17 67 ... K=3061 (prime)
... 81 -31 [50] 19 69 ... K=3089 (prime)
... 79 -29 [50] 21 71 ... K=3109 (prime)
... 77 -27 [50] 23 73 ... K=3121 (prime)
... 101 -50 [51] 1 52 ... K=2651 = 11 x 241
... 100 -49 [51] 2 53 ... K=2699 (prime)
... 98 -47 [51] 4 55 ... K=2789 (prime)
... 97 -46 [51] 5 56 ... K=2831 = 19 x 149
... 95 -44 [51] 7 58 ... K=2909 (prime)
... 94 -43 [51] 8 59 ... K=2945 = 5 x 19 x 31
... 92 -41 [51] 10 61 ... K=3011 (prime)
... 91 -40 [51] 11 62 ... K=3041 (prime)
... 89 -38 [51] 13 64 ... K=3095 = 5 x 619
... 88 -37 [51] 14 65 ... K=3119 (prime)
... 86 -35 [51] 16 67 ... K=3161 = 29 x 109
... 83 -32 [51] 19 70 ... K=3209 (prime)
... 82 -31 [51] 20 71 ... K=3221 (prime)
... 80 -29 [51] 22 73 ... K=3239 = 41 x 79
... 79 -28 [51] 23 74 ... K=3245 = 5 x 11 x 59
... 77 -26 [51] 25 76 ... K=3251 (prime)
... 103 -51 [52] 1 53 ... K=2755 = 5 x 19 x 29
... 101 -49 [52] 3 55 ... K=2851 (prime)
... 99 -47 [52] 5 57 ... K=2939 (prime)
... 97 -45 [52] 7 59 ... K=3019 (prime)
... 95 -43 [52] 9 61 ... K=3091 = 11 x 281
... 93 -41 [52] 11 63 ... K=3155 = 5 x 631
... 89 -37 [52] 15 67 ... K=3259 (prime)
... 87 -35 [52] 17 69 ... K=3299 (prime)
... 85 -33 [52] 19 71 ... K=3331 (prime)
... 83 -31 [52] 21 73 ... K=3355 = 5 x 11 x 61
... 81 -29 [52] 23 75 ... K=3371 (prime)
... 79 -27 [52] 25 77 ... K=3379 = 31 x 109
... 105 -52 [53] 1 54 ... K=2861 (prime)
... 104 -51 [53] 2 55 ... K=2911 = 41 x 71
... 103 -50 [53] 3 56 ... K=2959 = 11 x 269
... 102 -49 [53] 4 57 ... K=3005 = 5 x 601
... 101 -48 [53] 5 58 ... K=3049 (prime)
... 100 -47 [53] 6 59 ... K=3091 = 11 x 281
... 99 -46 [53] 7 60 ... K=3131 = 31 x 101
... 98 -45 [53] 8 61 ... K=3169 (prime)
... 97 -44 [53] 9 62 ... K=3205 = 5 x 641
... 96 -43 [53] 10 63 ... K=3239 = 41 x 79
... 95 -42 [53] 11 64 ... K=3271 (prime)
... 94 -41 [53] 12 65 ... K=3301 (prime)
... 93 -40 [53] 13 66 ... K=3329 (prime)
... 92 -39 [53] 14 67 ... K=3355 = 5 x 11 x 61
... 91 -38 [53] 15 68 ... K=3379 = 31 x 109
... 90 -37 [53] 16 69 ... K=3401 = 19 x 179
... 89 -36 [53] 17 70 ... K=3421 = 11 x 311
... 88 -35 [53] 18 71 ... K=3439 = 19 x 181
... 87 -34 [53] 19 72 ... K=3455 = 5 x 691
... 86 -33 [53] 20 73 ... K=3469 (prime)
... 85 -32 [53] 21 74 ... K=3481 = 59 x 59
... 84 -31 [53] 22 75 ... K=3491 (prime)
... 83 -30 [53] 23 76 ... K=3499 (prime)
... 82 -29 [53] 24 77 ... K=3505 = 5 x 701
... 81 -28 [53] 25 78 ... K=3509 = 11 x 11 x 29
... 80 -27 [53] 26 79 ... K=3511 (prime)
... 107 -53 [54] 1 55 ... K=2969 (prime)
... 103 -49 [54] 5 59 ... K=3161 = 29 x 109
... 101 -47 [54] 7 61 ... K=3245 = 5 x 11 x 59
... 97 -43 [54] 11 65 ... K=3389 (prime)
... 95 -41 [54] 13 67 ... K=3449 (prime)
... 91 -37 [54] 17 71 ... K=3545 = 5 x 709
... 89 -35 [54] 19 73 ... K=3581 (prime)
... 85 -31 [54] 23 77 ... K=3629 = 19 x 191
... 83 -29 [54] 25 79 ... K=3641 = 11 x 331
... 109 -54 [55] 1 56 ... K=3079 (prime)
... 108 -53 [55] 2 57 ... K=3131 = 31 x 101
... 107 -52 [55] 3 58 ... K=3181 (prime)
... 106 -51 [55] 4 59 ... K=3229 (prime)
... 104 -49 [55] 6 61 ... K=3319 (prime)
... 103 -48 [55] 7 62 ... K=3361 (prime)
... 102 -47 [55] 8 63 ... K=3401 = 19 x 179
... 101 -46 [55] 9 64 ... K=3439 = 19 x 181
... 98 -43 [55] 12 67 ... K=3541 (prime)
... 97 -42 [55] 13 68 ... K=3571 (prime)
... 96 -41 [55] 14 69 ... K=3599 = 59 x 61
... 94 -39 [55] 16 71 ... K=3649 = 41 x 89
... 93 -38 [55] 17 72 ... K=3671 (prime)
... 92 -37 [55] 18 73 ... K=3691 (prime)
... 91 -36 [55] 19 74 ... K=3709 (prime)
... 89 -34 [55] 21 76 ... K=3739 (prime)
... 87 -32 [55] 23 78 ... K=3761 (prime)
... 86 -31 [55] 24 79 ... K=3769 (prime)
... 84 -29 [55] 26 81 ... K=3779 (prime)
... 83 -28 [55] 27 82 ... K=3781 = 19 x 199
... 111 -55 [56] 1 57 ... K=3191 (prime)
... 109 -53 [56] 3 59 ... K=3295 = 5 x 659
... 107 -51 [56] 5 61 ... K=3391 (prime)
... 103 -47 [56] 9 65 ... K=3559 (prime)
... 101 -45 [56] 11 67 ... K=3631 (prime)
... 99 -43 [56] 13 69 ... K=3695 = 5 x 739
... ...

Appendix B

Factorisation of Cassini Constants

This is a list of Cassini constants in numerical order and, if not prime, their breakdown into the first prime factor (and its power) and the quotient (if any). As per Conjecture 2a, both the first factor and the quotient should appear as a constant earlier in the list, enabling a complete factorisation into primes.

1
5 is prime
11 is prime
19 is prime
29 is prime
31 is prime
41 is prime
55 = 5 x 11
59 is prime
61 is prime
71 is prime
79 is prime
89 is prime
95 = 5 x 19
101 is prime
109 is prime
121 = 11^2
131 is prime
139 is prime
145 = 5 x 29
149 is prime
151 is prime
155 = 5 x 31
179 is prime
181 is prime
191 is prime
199 is prime
205 = 5 x 41
209 = 11 x 19
211 is prime
229 is prime
239 is prime
241 is prime
251 is prime
269 is prime
271 is prime
281 is prime
295 = 5 x 59
305 = 5 x 61
311 is prime
319 = 11 x 29
331 is prime
341 = 11 x 31
349 is prime
355 = 5 x 71
359 is prime
361 = 19^2
379 is prime
389 is prime
395 = 5 x 79
401 is prime
409 is prime
419 is prime
421 is prime
431 is prime
439 is prime
445 = 5 x 89
449 is prime
451 = 11 x 41
461 is prime
479 is prime
491 is prime
499 is prime
505 = 5 x 101
509 is prime
521 is prime
541 is prime
545 = 5 x 109
551 = 19 x 29
569 is prime
571 is prime
589 = 19 x 31
599 is prime
601 is prime
605 = 5 x 121
619 is prime
631 is prime
641 is prime
649 = 11 x 59
655 = 5 x 131
659 is prime
661 is prime
671 = 11 x 61
691 is prime
695 = 5 x 139
701 is prime
709 is prime
719 is prime
739 is prime
745 = 5 x 149
751 is prime
755 = 5 x 151
761 is prime
769 is prime
779 = 19 x 41
781 = 11 x 71
809 is prime
811 is prime
821 is prime
829 is prime
839 is prime
841 = 29^2
859 is prime
869 = 11 x 79
881 is prime
895 = 5 x 179
899 = 29 x 31
905 = 5 x 181
911 is prime
919 is prime
929 is prime
941 is prime
955 = 5 x 191
961 = 31^2
971 is prime
979 = 11 x 89
991 is prime
995 = 5 x 199
1009 is prime
1019 is prime
1021 is prime
1031 is prime
1039 is prime
1045 = 5 x 209
1049 is prime
1051 is prime
1055 = 5 x 211
1061 is prime
1069 is prime
1091 is prime
1109 is prime
1111 = 11 x 101
1121 = 19 x 59
1129 is prime
1145 = 5 x 229
1151 is prime
1159 = 19 x 61
1171 is prime
1181 is prime
1189 = 29 x 41
1195 = 5 x 239
1199 = 11 x 109
1201 is prime
1205 = 5 x 241
1229 is prime
1231 is prime
1249 is prime
1255 = 5 x 251
1259 is prime
1271 = 31 x 41
1279 is prime
1289 is prime
1291 is prime
1301 is prime
1319 is prime
1321 is prime
1331 = 11^3
1345 = 5 x 269
1349 = 19 x 71
1355 = 5 x 271
1361 is prime
1381 is prime
1399 is prime
1405 = 5 x 281
1409 is prime
1429 is prime
1439 is prime
1441 = 11 x 131

1451 is prime
1459 is prime
1471 is prime
1481 is prime
1489 is prime
1499 is prime
1501 = 19 x 79
1511 is prime
1529 = 11 x 139
1531 is prime
1549 is prime
1555 = 5 x 311
1559 is prime
1571 is prime
1579 is prime
1595 = 5 x 319
1601 is prime
1609 is prime
1619 is prime
1621 is prime
1639 = 11 x 149
1655 = 5 x 331
1661 = 11 x 151
1669 is prime
1681 = 41^2
1691 = 19 x 89
1699 is prime
1705 = 5 x 341
1709 is prime
1711 = 29 x 59
1721 is prime
1741 is prime
1745 = 5 x 349
1759 is prime
1769 = 29 x 61
1789 is prime
1795 = 5 x 359
1801 is prime
1805 = 5 x 361
1811 is prime
1829 = 31 x 59
1831 is prime
1861 is prime
1871 is prime
1879 is prime
1889 is prime
1891 = 31 x 61
1895 = 5 x 379
1901 is prime
1919 = 19 x 101
1931 is prime
1945 = 5 x 389
1949 is prime
1951 is prime
1969 = 11 x 179
1979 is prime
1991 = 11 x 181
1999 is prime
2005 = 5 x 401
2011 is prime
2029 is prime
2039 is prime
2045 = 5 x 409
2059 = 29 x 71
2069 is prime
2071 = 19 x 109
2081 is prime
2089 is prime
2095 = 5 x 419
2099 is prime
2101 = 11 x 191
2105 = 5 x 421
2111 is prime
2129 is prime
2131 is prime
2141 is prime
2155 = 5 x 431
2161 is prime
2179 is prime
2189 = 11 x 199
2195 = 5 x 439
2201 = 31 x 71
2221 is prime
2239 is prime
2245 = 5 x 449
2251 is prime
2255 = 5 x 451
2269 is prime
2281 is prime
2291 = 29 x 79
2299 = 11^2 x 19
2305 = 5 x 461
2309 is prime
2311 is prime
2321 = 11 x 211
2339 is prime
2341 is prime
2351 is prime
2371 is prime
2381 is prime
2389 is prime
2395 = 5 x 479
2399 is prime
2411 is prime
2419 = 41 x 59
2441 is prime
2449 = 31 x 79
2455 = 5 x 491
2459 is prime
2489 = 19 x 131
2495 = 5 x 499
2501 = 41 x 61
2519 = 11 x 229
2521 is prime
2531 is prime
2539 is prime
2545 = 5 x 509
2549 is prime
2551 is prime
2579 is prime
2581 = 29 x 89
2591 is prime
2605 = 5 x 521
2609 is prime
2621 is prime
2629 = 11 x 239
2641 = 19 x 139
2651 = 11 x 241
2659 is prime
2671 is prime
2689 is prime
2699 is prime
2705 = 5 x 541
2711 is prime
2719 is prime
2729 is prime
2731 is prime
2741 is prime
2749 is prime
2755 = 5 x 551
2759 = 31 x 89
2761 = 11 x 251
2789 is prime
2791 is prime
2801 is prime
2819 is prime
2831 = 19 x 149
2845 = 5 x 569
2851 is prime
2855 = 5 x 571
2861 is prime
2869 = 19 x 151
2879 is prime
2909 is prime
2911 = 41 x 71
2929 = 29 x 101
2939 is prime
2945 = 5 x 589
2959 = 11 x 269
2969 is prime
2971 is prime
2981 = 11 x 271
2995 = 5 x 599
2999 is prime
3001 is prime
3005 = 5 x 601
3011 is prime
3019 is prime
3041 is prime
3049 is prime
3061 is prime
3079 is prime
3089 is prime
3091 = 11 x 281
3095 = 5 x 619
3109 is prime
3119 is prime
3121 is prime
3131 = 31 x 101
3155 = 5 x 631
3161 = 29 x 109
3169 is prime
3181 is prime
3191 is prime
3205 = 5 x 641
3209 is prime
3221 is prime
3229 is prime
3239 = 41 x 79
3245 = 5 x 649
3251 is prime
3259 is prime
3271 is prime
3295 = 5 x 659
3299 is prime
3301 is prime
3305 = 5 x 661
3319 is prime
3329 is prime
3331 is prime
3355 = 5 x 671
3359 is prime
3361 is prime
3371 is prime
3379 = 31 x 109
3389 is prime
3391 is prime
3401 = 19 x 179
3421 = 11 x 311
3439 = 19 x 181
3449 is prime
3455 = 5 x 691

3461 is prime
3469 is prime
3481 = 59^2
3491 is prime
3499 is prime
3505 = 5 x 701
3509 = 11^2 x 29
3511 is prime
3529 is prime
3539 is prime
3541 is prime
3545 = 5 x 709
3559 is prime
3571 is prime
3581 is prime
3595 = 5 x 719
3599 = 59 x 61
3629 = 19 x 191
3631 is prime
3641 = 11 x 331
3649 = 41 x 89
3659 is prime
3671 is prime
3691 is prime
3695 = 5 x 739
3701 is prime
3709 is prime
3719 is prime
3721 = 61^2
3739 is prime
3751 = 11^2 x 31
3755 = 5 x 751
3761 is prime
3769 is prime
3779 is prime
3781 = 19 x 199
3799 = 29 x 131
3805 = 5 x 761
3821 is prime
3839 = 11 x 349
3845 = 5 x 769
3851 is prime
3881 is prime
3889 is prime
3895 = 5 x 779
3905 = 5 x 781
3911 is prime
3919 is prime
3929 is prime
3931 is prime
3949 = 11 x 359
3971 = 11 x 361
3989 is prime
4001 is prime
4009 = 19 x 211
4019 is prime
4021 is prime
4031 = 29 x 139
4045 = 5 x 809
4049 is prime
4051 is prime
4055 = 5 x 811
4061 = 31 x 131
4079 is prime
4091 is prime
4099 is prime
4105 = 5 x 821
4111 is prime
4129 is prime
4139 is prime
4141 = 41 x 101
4145 = 5 x 829
4159 is prime
4169 = 11 x 379
4189 = 59 x 71
4195 = 5 x 839
4201 is prime
4205 = 5 x 841
4211 is prime
4219 is prime
4229 is prime
4231 is prime
4241 is prime
4259 is prime
4261 is prime
4271 is prime
4279 = 11 x 389
4289 is prime
4295 = 5 x 859
4309 = 31 x 139
4321 = 29 x 149
4331 = 61 x 71
4339 is prime
4345 = 5 x 869
4349 is prime
4351 = 19 x 229
4379 = 29 x 151
4391 is prime
4405 = 5 x 881
4409 is prime
4411 = 11 x 401
4421 is prime
4441 is prime
4451 is prime
4469 = 41 x 109
4481 is prime
4495 = 5 x 899
4499 = 11 x 409
4519 is prime
4541 = 19 x 239
4549 is prime
4555 = 5 x 911
4561 is prime
4579 = 19 x 241
4591 is prime
4595 = 5 x 919
4609 = 11 x 419
4619 = 31 x 149
4621 is prime
4631 = 11 x 421
4639 is prime
4645 = 5 x 929
4649 is prime
4651 is prime
4661 = 59 x 79
4679 is prime
4681 = 31 x 151
4691 is prime
4705 = 5 x 941
4721 is prime
4729 is prime
4741 = 11 x 431
4751 is prime
4759 is prime
4769 = 19 x 251
4789 is prime
4799 is prime
4801 is prime
4805 = 5 x 961
4819 = 61 x 79
4829 = 11 x 439
4831 is prime
4855 = 5 x 971
4861 is prime
4871 is prime
4889 is prime
4895 = 5 x 979
4909 is prime
4919 is prime
4931 is prime
4939 = 11 x 449
4951 is prime
4955 = 5 x 991
4961 = 11^2 x 41
4969 is prime
4999 is prime
5009 is prime
5011 is prime
5021 is prime
5039 is prime
5041 = 71^2
5045 = 5 x 1009
5051 is prime
5059 is prime
5071 = 11 x 461
5081 is prime
5095 = 5 x 1019
5099 is prime
5101 is prime
5105 = 5 x 1021
5111 = 19 x 269
5119 is prime
5149 = 19 x 271
5155 = 5 x 1031
5171 is prime
5179 is prime
5189 is prime
5191 = 29 x 179
5195 = 5 x 1039
5209 is prime
5231 is prime
5245 = 5 x 1049
5249 = 29 x 181
5251 = 59 x 89
5255 = 5 x 1051
5261 is prime
5269 = 11 x 479
5279 is prime
5281 is prime
5305 = 5 x 1061
5309 is prime
5339 = 19 x 281
5345 = 5 x 1069
5351 is prime
5371 = 41 x 131
5381 is prime
5399 is prime
5401 = 11 x 491
5419 is prime
5429 = 61 x 89
5431 is prime
5441 is prime
5449 is prime
5455 = 5 x 1091
5471 is prime
5479 is prime
5489 = 11 x 499
5501 is prime
5519 is prime
5521 is prime
5531 is prime
5539 = 29 x 191

5545 = 5 x 1109
5549 = 31 x 179
5555 = 5 x 1111
5569 is prime
5581 is prime
5591 is prime
5599 = 11 x 509
5605 = 5 x 1121
5609 = 71 x 79
5611 = 31 x 181
5639 is prime
5641 is prime
5645 = 5 x 1129
5651 is prime
5659 is prime
5669 is prime
5689 is prime
5699 = 41 x 139
5701 is prime
5711 is prime
5731 = 11 x 521
5741 is prime
5749 is prime
5755 = 5 x 1151
5771 = 29 x 199
5779 is prime
5791 is prime
5795 = 5 x 1159
5801 is prime
5821 is prime
5839 is prime
5849 is prime
5851 is prime
5855 = 5 x 1171
5861 is prime
5869 is prime
5879 is prime
5881 is prime
5905 = 5 x 1181
5909 = 19 x 311
5921 = 31 x 191
5939 is prime
5945 = 5 x 1189
5951 = 11 x 541
5959 = 59 x 101
5981 is prime
5995 = 5 x 1199
6005 = 5 x 1201
6011 is prime
6029 is prime
6061 = 11 x 551
6079 is prime
6089 is prime
6091 is prime
6101 is prime
6109 = 41 x 149
6119 = 29 x 211
6121 is prime
6131 is prime
6145 = 5 x 1229
6151 is prime
6155 = 5 x 1231
6161 = 61 x 101
6169 = 31 x 199
6191 = 41 x 151
6199 is prime
6211 is prime
6221 is prime
6229 is prime
6241 = 79^2
6245 = 5 x 1249
6259 = 11 x 569
6269 is prime
6271 is prime
6281 = 11 x 571
6289 = 19 x 331
6295 = 5 x 1259
6299 is prime
6301 is prime
6311 is prime
6319 = 71 x 89
6329 is prime
6355 = 5 x 1271
6359 is prime
6361 is prime
6379 is prime
6389 is prime
6395 = 5 x 1279
6421 is prime
6431 = 59 x 109
6445 = 5 x 1289
6449 is prime
6451 is prime
6455 = 5 x 1291
6469 is prime
6479 = 11 x 589
6481 is prime
6491 is prime
6505 = 5 x 1301
6521 is prime
6529 is prime
6541 = 31 x 211
6551 is prime
6569 is prime
6571 is prime
6581 is prime
6589 = 11 x 599
6595 = 5 x 1319
6599 is prime
6605 = 5 x 1321
6611 = 11 x 601
6619 is prime
6631 = 19 x 349
6641 = 29 x 229
6649 = 61 x 109
6655 = 5 x 1331
6659 is prime
6661 is prime
6679 is prime
6689 is prime
6691 is prime
6701 is prime
6709 is prime
6719 is prime
6745 = 5 x 1349
6761 is prime
6779 is prime
6781 is prime
6791 is prime
6805 = 5 x 1361
6809 = 11 x 619
6821 = 19 x 359
6829 is prime
6841 is prime
6859 = 19^3
6869 is prime
6871 is prime
6899 is prime
6905 = 5 x 1381
6911 is prime
6931 = 29 x 239
6941 = 11 x 631
6949 is prime
6959 is prime
6961 is prime
6971 is prime
6989 = 29 x 241
6991 is prime
6995 = 5 x 1399
7001 is prime
7019 is prime
7031 = 79 x 89
7039 is prime
7045 = 5 x 1409
7051 = 11 x 641
7069 is prime
7079 is prime
7099 = 31 x 229
7109 is prime
7121 is prime
7129 is prime
7139 = 11^2 x 59
7145 = 5 x 1429
7151 is prime
7159 is prime
7171 = 71 x 101
7195 = 5 x 1439
7201 = 19 x 379
7205 = 5 x 1441
7211 is prime
7219 is prime
7229 is prime
7249 = 11 x 659
7255 = 5 x 1451
7271 = 11 x 661
7279 = 29 x 251
7295 = 5 x 1459
7309 is prime
7321 is prime
7331 is prime
7339 = 41 x 179
7349 is prime
7351 is prime
7355 = 5 x 1471
7369 is prime
7381 = 11^2 x 61
7391 = 19 x 389
7405 = 5 x 1481
7409 = 31 x 239
7411 is prime
7421 = 41 x 181
7445 = 5 x 1489
7451 is prime
7459 is prime
7471 = 31 x 241
7481 is prime
7489 is prime
7495 = 5 x 1499
7499 is prime
7505 = 5 x 1501
7529 is prime
7541 is prime
7549 is prime
7555 = 5 x 1511
7559 is prime
7561 is prime
7589 is prime
7591 is prime
7601 = 11 x 691
7619 = 19 x 401
7621 is prime
… … …

Appendix C

Examples of Sequences Products

Here is a list of sequence products (p,q) × (r,s) for relatively prime sequences, where p>q (the complements rule can be used to obtain products for p<q). They are shown in ascending order of N=p+q+r+s; within that in descending order of p+q not less than r+s; then in descending order of p not less than q; and then all r,s in pairs (r,s) and (s,r) in descending order of r.
Where p+q=r+s we eliminate duplicates caused by p<r (commutative) or p<s (complementary). Where the resulting roots are not relatively prime, the relevant multiple is also shown, e.g. $K_{x,y}$: (p,q) × (r,s) = (x,y) = λ(u,v).
For each value of N, K has bounds: $5((N-2)^2) < K < (25N^4)/256 < (N^4)/10$.
For example, for N=100 the actual limits are 48,505 <= K <= 9,740,641.

####### N = 4 ####################
25 : (1,1) x (1,1) = (5,0) = 5(1,0)
####### N = 5 ####################
55 : (2,1) x (1,1) = (1,6)
####### N = 6 ####################
95 : (3,1) x (1,1) = (2,7)
121 : (2,1) x (2,1) = (3,7)
121 : (2,1) x (1,2) = (11,0) = 11(1,0)
####### N = 7 ####################
145 : (4,1) x (1,1) = (3,8)
155 : (3,2) x (1,1) = (1,11)
209 : (3,1) x (2,1) = (5,8)
209 : (3,1) x (1,2) = (1,13)
####### N = 8 ####################
205 : (5,1) x (1,1) = (4,9)
319 : (4,1) x (2,1) = (7,9)
319 : (4,1) x (1,2) = (2,15)
341 : (3,2) x (2,1) = (4,13)
341 : (3,2) x (1,2) = (17,1)
361 : (3,1) x (3,1) = (8,9)
361 : (3,1) x (1,3) = (19,0) = 19(1,0)
####### N = 9 ####################
275 : (6,1) x (1,1) = (5,10) = 5(1,2)
295 : (5,2) x (1,1) = (3,13)
305 : (4,3) x (1,1) = (1,16)
451 : (5,1) x (2,1) = (9,10)
451 : (5,1) x (1,2) = (3,17)
551 : (4,1) x (3,1) = (11,10)
551 : (4,1) x (1,3) = (1,22)
589 : (3,2) x (3,1) = (7,15)
589 : (3,2) x (1,3) = (20,3)
####### N = 10 ####################
355 : (7,1) x (1,1) = (6,11)
395 : (5,3) x (1,1) = (2,17)
605 : (6,1) x (2,1) = (11,11) = 11(1,1)
605 : (6,1) x (1,2) = (4,19)
649 : (5,2) x (2,1) = (8,15)
649 : (5,2) x (1,2) = (1,24)
671 : (4,3) x (2,1) = (5,19)
671 : (4,3) x (1,2) = (23,2)
779 : (5,1) x (3,1) = (14,11)
779 : (5,1) x (1,3) = (2,25)
841 : (4,1) x (4,1) = (15,11)
841 : (4,1) x (1,4) = (29,0) = 29(1,0)
899 : (4,1) x (3,2) = (10,17)
899 : (4,1) x (2,3) = (5,23)
961 : (3,2) x (3,2) = (5,24)
961 : (3,2) x (2,3) = (31,0) = 31(1,0)
####### N = 11 ####################
445 : (8,1) x (1,1) = (7,12)
475 : (7,2) x (1,1) = (5,15) = 5(1,3)
505 : (5,4) x (1,1) = (1,21)
781 : (7,1) x (2,1) = (13,12)
781 : (7,1) x (1,2) = (5,21)
869 : (5,3) x (2,1) = (7,20)
869 : (5,3) x (1,2) = (28,1)
1045 : (6,1) x (3,1) = (17,12)
1045 : (6,1) x (1,3) = (3,28)
1121 : (5,2) x (3,1) = (13,17)
1121 : (5,2) x (1,3) = (32,1)
1159 : (4,3) x (3,1) = (9,22)
1159 : (4,3) x (1,3) = (27,5)
1189 : (5,1) x (4,1) = (19,12)
1189 : (5,1) x (1,4) = (1,33)
1271 : (5,1) x (3,2) = (13,19)
1271 : (5,1) x (2,3) = (7,26)
####### N = 12 ####################
545 : (9,1) x (1,1) = (8,13)

605 : (7,3) x (1,1) = (4,19)
979 : (8,1) x (2,1) = (15,13)
979 : (8,1) x (1,2) = (6,23)
1045 : (7,2) x (2,1) = (12,17)
1045 : (7,2) x (1,2) = (3,28)
1111 : (5,4) x (2,1) = (6,25)
1111 : (5,4) x (1,2) = (29,3)
1349 : (7,1) x (3,1) = (20,13)
1349 : (7,1) x (1,3) = (4,31)
1501 : (5,3) x (3,1) = (12,23)
1501 : (5,3) x (1,3) = (33,4)
1595 : (6,1) x (4,1) = (23,13)
1595 : (6,1) x (1,4) = (2,37)
1705 : (6,1) x (3,2) = (16,21)
1705 : (6,1) x (2,3) = (9,29)
1711 : (5,2) x (4,1) = (18,19)
1711 : (5,2) x (1,4) = (37,3)
1829 : (5,2) x (3,2) = (11,28)
1829 : (5,2) x (2,3) = (4,37)
1769 : (4,3) x (4,1) = (13,25)
1769 : (4,3) x (1,4) = (31,8)
1891 : (4,3) x (3,2) = (6,35)
1891 : (4,3) x (2,3) = (42,1)
1681 : (5,1) x (5,1) = (24,13)
1681 : (5,1) x (1,5) = (41,0) = 41(1,0)
####### N = 13 ####################
655 : (10,1) x (1,1) = (9,14)
695 : (9,2) x (1,1) = (7,17)
725 : (8,3) x (1,1) = (5,20) = 5(1,4)
745 : (7,4) x (1,1) = (3,23)
755 : (6,5) x (1,1) = (1,26)
1199 : (9,1) x (2,1) = (17,14)
1199 : (9,1) x (1,2) = (7,25)
1331 : (7,3) x (2,1) = (11,22) = 11(1,2)
1331 : (7,3) x (1,2) = (1,35)
1691 : (8,1) x (3,1) = (23,14)
1691 : (8,1) x (1,3) = (5,34)
1805 : (7,2) x (3,1) = (19,19) = 19(1,1)
1805 : (7,2) x (1,3) = (1,41)
1919 : (5,4) x (3,1) = (11,29)
1919 : (5,4) x (1,3) = (34,7)
2059 : (7,1) x (4,1) = (27,14)
2059 : (7,1) x (1,4) = (3,41)
2201 : (7,1) x (3,2) = (19,23)
2201 : (7,1) x (2,3) = (11,32)
2291 : (5,3) x (4,1) = (17,26)
2291 : (5,3) x (1,4) = (38,7)
2449 : (5,3) x (3,2) = (9,37)
2449 : (5,3) x (2,3) = (1,48)
2255 : (6,1) x (5,1) = (29,14)
2255 : (6,1) x (1,5) = (1,46)
2419 : (5,2) x (5,1) = (23,21)
2419 : (5,2) x (1,5) = (42,5)
2501 : (4,3) x (5,1) = (17,28)
2501 : (4,3) x (1,5) = (35,11)
####### N = 14 ####################
775 : (11,1) x (1,1) = (10,15) = 5(2,3)
895 : (7,5) x (1,1) = (2,27)
1441 : (10,1) x (2,1) = (19,15)
1441 : (10,1) x (1,2) = (8,27)
1529 : (9,2) x (2,1) = (16,19)
1529 : (9,2) x (1,2) = (5,32)
1595 : (8,3) x (2,1) = (13,23)
1595 : (8,3) x (1,2) = (2,37)
1639 : (7,4) x (2,1) = (10,27)
1639 : (7,4) x (1,2) = (39,1)
1661 : (6,5) x (2,1) = (7,31)
1661 : (6,5) x (1,2) = (35,4)
2071 : (9,1) x (3,1) = (26,15)
2071 : (9,1) x (1,3) = (6,37)
2299 : (7,3) x (3,1) = (18,25)
2299 : (7,3) x (1,3) = (45,2)
2581 : (8,1) x (4,1) = (31,15)
2581 : (8,1) x (1,4) = (4,45)
2759 : (8,1) x (3,2) = (22,25)
2759 : (8,1) x (2,3) = (13,35)
2755 : (7,2) x (4,1) = (26,21)
2755 : (7,2) x (1,4) = (51,1)
2945 : (7,2) x (3,2) = (17,32)
2945 : (7,2) x (2,3) = (8,43)
2929 : (5,4) x (4,1) = (16,33)
2929 : (5,4) x (1,4) = (39,11)
3131 : (5,4) x (3,2) = (7,46)
3131 : (5,4) x (2,3) = (53,2)
2911 : (7,1) x (5,1) = (34,15)
2911 : (7,1) x (1,5) = (2,51)
3239 : (5,3) x (5,1) = (22,29)
3239 : (5,3) x (1,5) = (43,10)
3025 : (6,1) x (6,1) = (35,15) = 5(7,3)
3025 : (6,1) x (1,6) = (55,0) = 55(1,0)
3245 : (6,1) x (5,2) = (28,23)
3245 : (6,1) x (2,5) = (7,47)
3355 : (6,1) x (4,3) = (21,31)
3355 : (6,1) x (3,4) = (14,39)
3481 : (5,2) x (5,2) = (21,32)
3481 : (5,2) x (2,5) = (59,0) = 59(1,0)
3599 : (5,2) x (4,3) = (14,41)
3599 : (5,2) x (3,4) = (7,50)
3721 : (4,3) x (4,3) = (7,51)
3721 : (4,3) x (3,4) = (61,0) = 61(1,0)
####### N = 15 ###################
905 : (12,1) x (1,1) = (11,16)
955 : (11,2) x (1,1) = (9,19)
995 : (10,3) x (1,1) = (7,22)
1025 : (9,4) x (1,1) = (5,25) = 5(1,5)
1045 : (8,5) x (1,1) = (3,28)

1055 : (7,6) x (1,1) = (1,31)
1705 : (11,1) x (2,1) = (21,16)
1705 : (11,1) x (1,2) = (9,29)
1969 : (7,5) x (2,1) = (9,32)
1969 : (7,5) x (1,2) = (40,3)
2489 : (10,1) x (3,1) = (29,16)
2489 : (10,1) x (1,3) = (7,40)
2641 : (9,2) x (3,1) = (25,21)
2641 : (9,2) x (1,3) = (3,47)
2755 : (8,3) x (3,1) = (21,26)
2755 : (8,3) x (1,3) = (51,1)
2831 : (7,4) x (3,1) = (17,31)
2831 : (7,4) x (1,3) = (46,5)
2869 : (6,5) x (3,1) = (13,36)
2869 : (6,5) x (1,3) = (41,9)
3161 : (9,1) x (4,1) = (35,16)
3161 : (9,1) x (1,4) = (5,49)
3379 : (9,1) x (3,2) = (25,27)
3379 : (9,1) x (2,3) = (15,38)
3509 : (7,3) x (4,1) = (25,28)
3509 : (7,3) x (1,4) = (52,5)
3751 : (7,3) x (3,2) = (15,41)
3751 : (7,3) x (2,3) = (5,54)
3649 : (8,1) x (5,1) = (39,16)
3649 : (8,1) x (1,5) = (3,56)
3895 : (7,2) x (5,1) = (33,23)
3895 : (7,2) x (1,5) = (58,3)
4141 : (5,4) x (5,1) = (21,37)
4141 : (5,4) x (1,5) = (44,15)
3905 : (7,1) x (6,1) = (41,16)
3905 : (7,1) x (1,6) = (1,61)
4189 : (7,1) x (5,2) = (33,25)
4189 : (7,1) x (2,5) = (9,52)
4331 : (7,1) x (4,3) = (25,34)
4331 : (7,1) x (3,4) = (17,43)
4345 : (5,3) x (6,1) = (27,32)
4345 : (5,3) x (1,6) = (48,13)
4661 : (5,3) x (5,2) = (19,43)
4661 : (5,3) x (2,5) = (61,5)
4819 : (5,3) x (4,3) = (11,54)
4819 : (5,3) x (3,4) = (3,65)
####### N = 16 ####################
1045 : (13,1) x (1,1) = (12,17)
1145 : (11,3) x (1,1) = (8,23)
1205 : (9,5) x (1,1) = (4,29)
1991 : (12,1) x (2,1) = (23,17)
1991 : (12,1) x (1,2) = (10,31)
2101 : (11,2) x (2,1) = (20,21)
2101 : (11,2) x (1,2) = (7,36)
2189 : (10,3) x (2,1) = (17,25)
2189 : (10,3) x (1,2) = (4,41)
2255 : (9,4) x (2,1) = (14,29)
2255 : (9,4) x (1,2) = (1,46)
2299 : (8,5) x (2,1) = (11,33) = 11(1,3)
2299 : (8,5) x (1,2) = (45,2)
2321 : (7,6) x (2,1) = (8,37)
2321 : (7,6) x (1,2) = (41,5)
2945 : (11,1) x (3,1) = (32,17)
2945 : (11,1) x (1,3) = (8,43)
3401 : (7,5) x (3,1) = (16,37)
3401 : (7,5) x (1,3) = (47,8)
3799 : (10,1) x (4,1) = (39,17)
3799 : (10,1) x (1,4) = (6,53)
4061 : (10,1) x (3,2) = (28,29)
4061 : (10,1) x (2,3) = (17,41)
4031 : (9,2) x (4,1) = (34,23)
4031 : (9,2) x (1,4) = (1,62)
4309 : (9,2) x (3,2) = (23,36)
4309 : (9,2) x (2,3) = (12,49)
4205 : (8,3) x (4,1) = (29,29) = 29(1,1)
4205 : (8,3) x (1,4) = (59,4)
4495 : (8,3) x (3,2) = (18,43)
4495 : (8,3) x (2,3) = (7,57)
4321 : (7,4) x (4,1) = (24,35)
4321 : (7,4) x (1,4) = (53,9)
4619 : (7,4) x (3,2) = (13,50)
4619 : (7,4) x (2,3) = (2,65)
4379 : (6,5) x (4,1) = (19,41)
4379 : (6,5) x (1,4) = (47,14)
4681 : (6,5) x (3,2) = (8,57)
4681 : (6,5) x (2,3) = (64,3)
4469 : (9,1) x (5,1) = (44,17)
4469 : (9,1) x (1,5) = (4,61)
4961 : (7,3) x (5,1) = (32,31)
4961 : (7,3) x (1,5) = (59,8)
4895 : (8,1) x (6,1) = (47,17)
4895 : (8,1) x (1,6) = (2,67)
5251 : (8,1) x (5,2) = (38,27)
5251 : (8,1) x (2,5) = (11,57)
5429 : (8,1) x (4,3) = (29,37)
5429 : (8,1) x (3,4) = (20,47)
5225 : (7,2) x (6,1) = (40,25) = 5(8,5)
5225 : (7,2) x (1,6) = (65,5) = 5(13,1)
5605 : (7,2) x (5,2) = (31,36)
5605 : (7,2) x (2,5) = (4,69)
5795 : (7,2) x (4,3) = (22,47)
5795 : (7,2) x (3,4) = (13,58)
5555 : (5,4) x (6,1) = (26,41)
5555 : (5,4) x (1,6) = (49,19)
5959 : (5,4) x (5,2) = (17,54)
5959 : (5,4) x (2,5) = (63,10)
6161 : (5,4) x (4,3) = (8,67)
6161 : (5,4) x (3,4) = (77,1)
5041 : (7,1) x (7,1) = (48,17)
5041 : (7,1) x (1,7) = (71,0) = 71(1,0)
5609 : (7,1) x (5,3) = (32,35)

5609 : (7,1) x (3,5) = (16,53)
6241 : (5,3) x (5,3) = (16,57)
6241 : (5,3) x (3,5) = (79,0) = 79(1,0)
####### N = 17 ####################
1195 : (14,1) x (1,1) = (13,18)
1255 : (13,2) x (1,1) = (11,21)
1345 : (11,4) x (1,1) = (7,27)
1405 : (8,7) x (1,1) = (1,36)
2299 : (13,1) x (2,1) = (25,18)
2299 : (13,1) x (1,2) = (11,33) = 11(1,3)
2519 : (11,3) x (2,1) = (19,26)
2519 : (11,3) x (1,2) = (5,43)
2651 : (9,5) x (2,1) = (13,34)
2651 : (9,5) x (1,2) = (50,1)
3439 : (12,1) x (3,1) = (35,18)
3439 : (12,1) x (1,3) = (9,46)
3629 : (11,2) x (3,1) = (31,23)
3629 : (11,2) x (1,3) = (5,53)
3781 : (10,3) x (3,1) = (27,28)
3781 : (10,3) x (1,3) = (1,60)
3895 : (9,4) x (3,1) = (23,33)
3895 : (9,4) x (1,3) = (58,3)
3971 : (8,5) x (3,1) = (19,38) = 19(1,2)
3971 : (8,5) x (1,3) = (53,7)
4009 : (7,6) x (3,1) = (15,43)
4009 : (7,6) x (1,3) = (48,11)
4495 : (11,1) x (4,1) = (43,18)
4495 : (11,1) x (1,4) = (7,57)
4805 : (11,1) x (3,2) = (31,31) = 31(1,1)
4805 : (11,1) x (2,3) = (19,44)
5191 : (7,5) x (4,1) = (23,42)
5191 : (7,5) x (1,4) = (54,13)
5549 : (7,5) x (3,2) = (11,59)
5549 : (7,5) x (2,3) = (73,1)
5371 : (10,1) x (5,1) = (49,18)
5371 : (10,1) x (1,5) = (5,66)
5699 : (9,2) x (5,1) = (43,25)
5699 : (9,2) x (1,5) = (74,1)
5945 : (8,3) x (5,1) = (37,32)
5945 : (8,3) x (1,5) = (67,7)
6109 : (7,4) x (5,1) = (31,39)
6109 : (7,4) x (1,5) = (60,13)
6191 : (6,5) x (5,1) = (25,46)
6191 : (6,5) x (1,5) = (53,19)
5995 : (9,1) x (6,1) = (53,18)
5995 : (9,1) x (1,6) = (3,73)
6431 : (9,1) x (5,2) = (43,29)
6431 : (9,1) x (2,5) = (13,62)
6649 : (9,1) x (4,3) = (33,40)
6649 : (9,1) x (3,4) = (23,51)
6655 : (7,3) x (6,1) = (39,34)
6655 : (7,3) x (1,6) = (66,11) = 11(6,1)
7139 : (7,3) x (5,2) = (29,47)
7139 : (7,3) x (2,5) = (83,1)
7381 : (7,3) x (4,3) = (19,60)
7381 : (7,3) x (3,4) = (9,73)
6319 : (8,1) x (7,1) = (55,18)
6319 : (8,1) x (1,7) = (1,78)
7031 : (8,1) x (5,3) = (37,38)
7031 : (8,1) x (3,5) = (19,58)
6745 : (7,2) x (7,1) = (47,27)
6745 : (7,2) x (1,7) = (72,7)
7505 : (7,2) x (5,3) = (29,49)
7505 : (7,2) x (3,5) = (11,71)
7171 : (5,4) x (7,1) = (31,45)
7171 : (5,4) x (1,7) = (54,23)
7979 : (5,4) x (5,3) = (13,71)
7979 : (5,4) x (3,5) = (82,5)
####### N = 18 ####################
1355 : (15,1) x (1,1) = (14,19)
1475 : (13,3) x (1,1) = (10,25) = 5(2,5)
1555 : (11,5) x (1,1) = (6,31)
1595 : (9,7) x (1,1) = (2,37)
2629 : (14,1) x (2,1) = (27,19)
2629 : (14,1) x (1,2) = (12,35)
2761 : (13,2) x (2,1) = (24,23)
2761 : (13,2) x (1,2) = (9,40)
2959 : (11,4) x (2,1) = (18,31)
2959 : (11,4) x (1,2) = (3,50)
3091 : (8,7) x (2,1) = (9,43)
3091 : (8,7) x (1,2) = (47,6)
3971 : (13,1) x (3,1) = (38,19) = 19(2,1)
3971 : (13,1) x (1,3) = (10,49)
4351 : (11,3) x (3,1) = (30,29)
4351 : (11,3) x (1,3) = (2,63)
4579 : (9,5) x (3,1) = (22,39)
4579 : (9,5) x (1,3) = (59,6)
5249 : (12,1) x (4,1) = (47,19)
5249 : (12,1) x (1,4) = (8,61)
5611 : (12,1) x (3,2) = (34,33)
5611 : (12,1) x (2,3) = (21,47)
5539 : (11,2) x (4,1) = (42,25)
5539 : (11,2) x (1,4) = (3,70)
5921 : (11,2) x (3,2) = (29,40)
5921 : (11,2) x (2,3) = (16,55)
5771 : (10,3) x (4,1) = (37,31)
5771 : (10,3) x (1,4) = (73,2)
6169 : (10,3) x (3,2) = (24,47)
6169 : (10,3) x (2,3) = (11,63)
5945 : (9,4) x (4,1) = (32,37)
5945 : (9,4) x (1,4) = (67,7)
6355 : (9,4) x (3,2) = (19,54)
6355 : (9,4) x (2,3) = (6,71)
6061 : (8,5) x (4,1) = (27,43)
6061 : (8,5) x (1,4) = (61,12)
6479 : (8,5) x (3,2) = (14,61)

6479 : (8,5) x (2,3) = (1,79)
6479 : (8,5) x (2,3) = (1,79)
6119 : (7,6) x (4,1) = (22,49)
6119 : (7,6) x (1,4) = (55,17)
6541 : (7,6) x (3,2) = (9,68)
6541 : (7,6) x (2,3) = (75,4)
6355 : (11,1) x (5,1) = (54,19)
6355 : (11,1) x (1,5) = (6,71)
7339 : (7,5) x (5,1) = (30,47)
7339 : (7,5) x (1,5) = (61,18)
7205 : (10,1) x (6,1) = (59,19)
7205 : (10,1) x (1,6) = (4,79)
7729 : (10,1) x (5,2) = (48,31)
7729 : (10,1) x (2,5) = (15,67)
7991 : (10,1) x (4,3) = (37,43)
7991 : (10,1) x (3,4) = (26,55)
7645 : (9,2) x (6,1) = (52,27)
7645 : (9,2) x (1,6) = (83,3)
8201 : (9,2) x (5,2) = (41,40)
8201 : (9,2) x (2,5) = (8,79)
8479 : (9,2) x (4,3) = (30,53)
8479 : (9,2) x (3,4) = (19,66)
7975 : (8,3) x (6,1) = (45,35) = 5(9,7)
7975 : (8,3) x (1,6) = (75,10) = 5(15,2)
8555 : (8,3) x (5,2) = (34,49)
8555 : (8,3) x (2,5) = (1,91)
8845 : (8,3) x (4,3) = (23,63)
8845 : (8,3) x (3,4) = (12,77)
8195 : (7,4) x (6,1) = (38,43)
8195 : (7,4) x (1,6) = (67,17)
8791 : (7,4) x (5,2) = (27,58)
8791 : (7,4) x (2,5) = (85,6)
9089 : (7,4) x (4,3) = (16,73)
9089 : (7,4) x (3,4) = (5,88)
8305 : (6,5) x (6,1) = (31,51)
8305 : (6,5) x (1,6) = (59,24)
8909 : (6,5) x (5,2) = (20,67)
8909 : (6,5) x (2,5) = (76,13)
9211 : (6,5) x (4,3) = (9,83)
9211 : (6,5) x (3,4) = (93,2)
7739 : (9,1) x (7,1) = (62,19)
7739 : (9,1) x (1,7) = (2,85)
8611 : (9,1) x (5,3) = (42,41)
8611 : (9,1) x (3,5) = (22,63)
8591 : (7,3) x (7,1) = (46,37)
8591 : (7,3) x (1,7) = (73,14)
9559 : (7,3) x (5,3) = (26,63)
9559 : (7,3) x (3,5) = (6,89)
7921 : (8,1) x (8,1) = (63,19)
7921 : (8,1) x (1,8) = (89,0) = 89(1,0)
8455 : (8,1) x (7,2) = (54,29)
8455 : (8,1) x (2,7) = (9,79)
8989 : (8,1) x (5,4) = (36,49)
8989 : (8,1) x (4,5) = (27,59)
9025 : (7,2) x (7,2) = (45,40) = 5(9,8)
9025 : (7,2) x (2,7) = (95,0) = 95(1,0)
9595 : (7,2) x (5,4) = (27,62)
9595 : (7,2) x (4,5) = (18,73)
10201 : (5,4) x (5,4) = (9,88)
10201 : (5,4) x (4,5) = (101,0) = 101(1,0)
####### N = 19 ###################
1525 : (16,1) x (1,1) = (15,20) = 5(3,4)
1595 : (15,2) x (1,1) = (13,23)
1655 : (14,3) x (1,1) = (11,26)
1705 : (13,4) x (1,1) = (9,29)
1745 : (12,5) x (1,1) = (7,32)
1775 : (11,6) x (1,1) = (5,35) = 5(1,7)
1795 : (10,7) x (1,1) = (3,38)
1805 : (9,8) x (1,1) = (1,41)
2981 : (15,1) x (2,1) = (29,20)
2981 : (15,1) x (1,2) = (13,37)
3245 : (13,3) x (2,1) = (23,28)
3245 : (13,3) x (1,2) = (7,47)
3421 : (11,5) x (2,1) = (17,36)
3421 : (11,5) x (1,2) = (1,57)
3509 : (9,7) x (2,1) = (11,44) = 11(1,4)
3509 : (9,7) x (1,2) = (52,5)
4541 : (14,1) x (3,1) = (41,20)
4541 : (14,1) x (1,3) = (11,52)
4769 : (13,2) x (3,1) = (37,25)
4769 : (13,2) x (1,3) = (7,59)
5111 : (11,4) x (3,1) = (29,35)
5111 : (11,4) x (1,3) = (70,1)
5339 : (8,7) x (3,1) = (17,50)
5339 : (8,7) x (1,3) = (55,13)
6061 : (13,1) x (4,1) = (51,20)
6061 : (13,1) x (1,4) = (9,65)
6479 : (13,1) x (3,2) = (37,35)
6479 : (13,1) x (2,3) = (23,50)
6641 : (11,3) x (4,1) = (41,32)
6641 : (11,3) x (1,4) = (80,1)
7099 : (11,3) x (3,2) = (27,49)
7099 : (11,3) x (2,3) = (13,66)
6989 : (9,5) x (4,1) = (31,44)
6989 : (9,5) x (1,4) = (68,11)
7471 : (9,5) x (3,2) = (17,63)
7471 : (9,5) x (2,3) = (3,82)
7421 : (12,1) x (5,1) = (59,20)
7421 : (12,1) x (1,5) = (7,76)
7831 : (11,2) x (5,1) = (53,27)
7831 : (11,2) x (1,5) = (1,87)
8159 : (10,3) x (5,1) = (47,34)
8159 : (10,3) x (1,5) = (83,5)
8405 : (9,4) x (5,1) = (41,41) = 41(1,1)
8405 : (9,4) x (1,5) = (76,11)
… … …