Fixed-point-free variations
(Reference A336246)

V(n,k) is the number of ways to place n persons on different seats such that each person number p, 1≤p≤n differs from the seat number s(p), 1≤s(p)≤n+k, k≥0.

Annotation: Here, n is used rather than i; k is the number of free seats.
Abbreviation: V(n,k) is the number of (n,k)-arrangements (variations).

<u>Recurrence</u>: V(n,k)=(n+k-1)·V(n-1,k)+(n-1)·V(n-2,k) for n≥2, k≥0
with V(0,k)=1 and V(1,k)=k
For n=0, there is one empty arrangement. For n=1, the person can be placed on seat number 2 thru k+1 (if k>0).

<u>Explicite formula</u>: $V(n,k) = \sum_{r=0}^{n}(-1)^r \binom{n}{r}\frac{(n+k-r)!}{k!}$

A constructive proof of the recurrence is based on this fact:
For n≥2, any (n,k)-arrangement can be reduced to a (n-2,k)-arrangement by removing two persons and two seats or to a (n-1,k)-arrangement by removing one person and one seat. Inversely, any (n,k)-arrangement can be produced by extending (n-1,k)- and (n-2,k)-arrangements, see appendix.

<u>1. Proof of the recurrence</u>
There are two types of (n,k)-arrangements:
Type 1: s(n)<n and s(s(n))=n+k, type 2: s(n)>n or s(s(n))≠n+k
Let $V_j(n,k)$ be the number of (n,k)-arrangements of type j, j=1,2

Type 1
Person n selects the seat q=s(n), 1<=q<n, and person q selects the seat n+k. First we remove the persons and seats with numbers n and q . Then we reduce p→p'=p-1 for p>q and s'=s-1 for s>q. If p and s are both smaller or greater than q, then s'≠p' follows from s≠p. Otherwise |p-s|>1, as seat q is gone, and therefore |p'-s'|>0.
The inversion with given q, 1≤q<n, is unambiguous: Increase, in the (n–2,k)-arrangement, all seat numbers s≥q by 1 such that seat q does not occur and the largest possible seat number is n+k–1. Insert person q with seat n+k and person n with seat q. Therefore, the number of (n,k)-arrangements of type 1, with given q, equals the number of (n-2,k)-arrangements.
Summing over n-1 values of q, we yield $V_1(n,k)$=(n-1)·V(n-2,k).

Type 2
Person n selects the seat q=s(n), 1≤q≤n+k, q≠n, and, if q<n, person q selects the seat s(q)≠ n+k. First we remove person n so that seat q is free now. If seat n+k is also free, we remove this seat immediately. Otherwise, we first move the person p sitting there to seat q. This way, we yield a (n-1,k)-arrangement. Again the inversion, with given q, is unambiguous: If seat q is occupied in the (n-1,k)-arrangement, we move the person p≠q to seat n+k (which satisfies s(q)≠ n+k) and place person n on seat q.
Therefore, the number of (n,k)-arrangements of type 2, with given q, equals the number of (n-1,k)-arrangements.
Summing over n+k-1 values of q, we yield V_2(n,k)=(n+k-1)·V(n-2,k).

With V(n,k)= V_1(n,k)+ V_2(n,k), the recurrence is proved.

2. Proof of the explicite formula
Here the more general term 'variation' is used instead of 'seat arrangement'. For V(n,0), see 'Counting derangements' in
https://en.wikipedia.org/wiki/Inclusion%E2%80%93exclusion_principle

The number of variations of placing n persons on n+k seats is $\frac{(n+k)!}{k!}$.
Let A(q) be the set of variations with fixed point q (s(q)=q). Select r persons 1≤q_1< q_2<…< q_r≤n. The cardinality of $|A(q_1) \cap \ldots \cap A(q_r)|$ is $\frac{(n+k-r)!}{k!}$ because r points (persons and seats) are fixed. The number of selections of q_1, q_2,…, q_r is $\binom{n}{r}$ and, therefore, the number of variations with r fixed points is $\binom{n}{r}\frac{(n+k-r)!}{k!}$, multiple counts included. These are discounted by using the principle of inclusion-exclusion (see link above), which yields the number of all variations with at least one fixed point:
$\bar{V}(n,k) = \sum_{r=1}^{n}(-1)^{r-1}\binom{n}{r}\frac{(n+k-r)!}{k!}$.
Therefore, $V(n,k) = \frac{(n+k)!}{k!} - \bar{V}(n,k) = \sum_{r=0}^{n}(-1)^{r}\binom{n}{r}\frac{(n+k-r)!}{k!}$ q.e.d.

3. Appendix: Maxima code and output, using the recurrence

All variations V(n,k) with n persons and n+k seats are returned:

```
block(k:1, n:3, v2:[""],v3: [],
 for n0 from 0 thru n-1 do
  (n1:n0+1,n2: n0+2,v1:v2,v2:v3,v3:[],
  for nu from 1 thru length(v1) do  /*make type 2*/
     (for q from 1 thru n1+k do
       if q#n1 then
        (w:v1[nu], w:ssubst(string(n1+k),string(q),w),
        w:concat(w,string(q)),v2:append(v2,[w])),
     if n0<n-1 then
      for q from 1 thru n2-1 do   /*make type 1*/
       (w:v1[nu], u:string(q),
        for s from 1 thru n2-2 do
         (t:cint(substring(w,s,s+1)),
          if t>=48+q then w:ssubst(ascii(t+1),ascii(t),w,s,s+1)),
          w: concat(sinsert(string(n2+k),w,q),string(q)),
          v3: append(v3,[w])))),
    print(concat("V(",n,",",k,")","=",length(v2), " variations:")),
    return(v2) );
```

$$V(3,1) = 11\ variations:$$

[431,342,341,312,314,241,412,214,231,432,234]

The first triple means: Person 1 on seat 4, person 2 on seat 3, person 3 on seat 1.

$$V(4,1) = 53\ variations:$$

**[5421,4512,4153,5321,3512,2153,5341,3542,2453,4351,4312,4513,4315,
3421,3452,5423,3425,3451,3412,5413,3415,3521,3152,5123,3125,3541,
3142,5143,3145,2451,5412,2413,2415,4521,4152,4123,4125,2541,5142,
2143,2145,2351,5312,2513,2315,4321,4352,4523,4325,2341,5342,2543,
2345]**

$$V(3,2) = 32\ variations:$$

**[531,352,541,452,451,412,514,415,251,512,214,215,231,532,234,235,
241,542,254,245,351,312,314,315,431,432,534,435,341,342,354,345]**

$$V(4,2) = 181\ variations:$$

**[6521,5612,5163,6321,3612,2163,6341,3642,2463,6351,3652,2563,6421,
4612,4163,6541,5642,5463,6451,4652,4563,5361,5312,5613,6315,5316,
3521,3562,6523,3625,3526,5461,5412,5413,6415,5416,4521,4562,4523,
4625,4526,4561,4512,4513,4615,4516,4621,4162,4123,4125,4126,5641,
5142,5143,6145,5146,4651,4152,4153,4165,4156,2561,6512,2513,2615,
2516,5621,5162,5123,6125,5126,2641,6142,2143,2145,2146,2651,6152,
2153,2165,2156,2361,6312,2613,2315,2316,5321,5362,5623,6325,5326,
2341,6342,2643,2345,2346,2351,6352,2653,2365,2356,2461,6412,2413,
2415,2416,5421,5462,5423,6425,5426,2541,6542,2543,2645,2546,2451,
6452,2453,2465,2456,3561,3512,6513,3615,3516,3621,3162,6123,3125,
3126,3641,3142,6143,3145,3146,3651,3152,6153,3165,3156,4361,4312,
4613,4315,4316,4321,4362,4623,4325,4326,5341,5342,5643,6345,5346,
4351,4352,4653,4365,4356,3461,3412,6413,3415,3416,3421,3462,6423,
3425,3426,3541,3542,6543,3645,3546,3451,3452,6453,3465,3456]**