Proofs of two observations from Sequence A334614 (*Ya-Ping Lu, Sep 22, 2020*)

NAME of A334614

a(n) = pi(prime(n) - n) + n, where pi is the prime counting function.

DATA of A334614

1, 2, 4, 6, 8, 10, 11, 13, 15, 18, 19, 21, 22, 24, 26, 28, 30, 32, 34, 35, 36, 38, 40, 42, 45, 47, 48, 50, 51, 53, 55, 57, 60, 61, 65, 66, 67, 68, 70, 72, 74, 76, 77, 79, 81, 82, 85, 88, 89, 91, 93, 94, 95, 99, 101, 102, 104, 105, 106, 107, 108, 112, 116, 117

Observation 1: A334614 is an increasing sequence, or a(n) > a(n-1) >= 1.

Proof: Let p_n be the n-th prime. By the definition of Sequence A334614,

$$a_n - a_{n-1} = [\pi(p_n - n) + n] - [\pi(p_{n-1} - (n-1)) + (n-1)]$$

or

$$a_n - a_{n-1} = \pi(p_n - n) - \pi(p_{n-1} - n + 1) + 1$$

The difference between the two arguments of the prime counting function, $p_n - n$ and $p_{n-1} - n + 1$, in the equation above is

$$(p_n - n) - (p_{n-1} - n + 1) = p_n - p_{n-1} - 1$$

in which $p_n - p_{n-1} \geq 1$ and thus $(p_n - n) - (p_{n-1} - n + 1) \geq 0$, or

$$p_n - n \geq p_{n-1} - n + 1.$$

Since prime counting function is non-decreasing, we have

$$\pi(p_n - n) \geq \pi(p_{n-1} - n + 1)$$

and

$$a_n - a_{n-1} = \pi(p_n - n) - \pi(p_{n-1} - n + 1) + 1 \geq 1.$$

Therefore

$$a_n > a_{n-1} > \cdots > a_1 = 1.$$

Observation 2: a(n) < 2*n.

Proof: Let k be the number of prime numbers less than or equal to $p_n - \mathrm{n}$, or

$$k = \pi(p_n - \mathrm{n})$$

where n ≥ 2. Since

$$p_k \leq p_n - \mathrm{n} < p_n$$

which means $p_k < p_n$ and k < n. So, we have

$$p_k + k \leq p_n - n + k < p_n$$

Lu and Deng [1] showed that there is at least one prime number in the range of $(p_k, p_k + k]$. By the definition of k, there is no prime number in the range of $(p_k, p_n - n]$. Thus, there must be at least one prime number in the range of $(p_n - n, p_k + k]$. Since $p_k + k < p_n$, there must be at least one prime number in the range of $(p_n - n, p_n)$, where n ≥ 2. As p_n is a prime number, there should be at least two prime numbers in the range of $(p_n - n, p_n]$, meaning

$$\pi(p_n) - \pi(p_n - n) \geq 2$$

or

$$\pi(p_n - n) \leq \pi(p_n) - 2 = n - 2.$$

By definition, $a_n = \pi(p_n - n) + n$. Since $\pi(p_n - n) \leq n - 2$, we have

$$a_n \leq n - 2 + n = 2n - 2$$

or, for n ≥ 2,

$$a_n \leq 2(n - 1) < 2n - 1 < 2n.$$

Reference

[1] Ya-Ping Lu and Shu-Fang Deng, An upper bound for the prime gap, arXiv:2007.15282 [math.GM], 2020.