The Bowl's Sequence

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1 The Power Series of $\arctan\left(1-\sqrt{1-x^2}\right)$ and Bowl's Sequence.

I have discussed about a function and its power series at $x = 0$. Here is the expression of the function. Obviously, $f(x)$ is an even function, only the even powers of x is contained in the series.

$$\begin{aligned} f(x) &= \arctan\left(1-\sqrt{1-x^2}\right) \\ &= \sum_{n=1}^{\infty} \frac{a_n}{(2n)!} x^{2n} \\ a_n &= 1, 3, 15, 315, 36855, 4833675, \\ & 711485775, 133449190875, \\ & 33399969978375, 10845524928112875, \\ & 4368604540935009375, \\ & 2121018409773134746875, \cdots \end{aligned}$$

I name the a_n Bowl's sequence, for the curve of $f(x)$ looks like a bowl. Now I try to get the recurrence formula. Firstly, note that

$$\begin{aligned} \sqrt{1-x^2} &= 1 - \sum_{n=0}^{\infty} \frac{\prod_{k=1}^{n-1}\left(k-\frac{1}{2}\right)}{2n!} x^{2n} \\ & 1 - \sum_{n=1}^{\infty} \frac{(2n-2)!}{(n!)(n-1)!2^{2n-1}} x^{2n} \\ (\arctan x)' &= \frac{1}{1+x^2} \end{aligned}$$

Then one can write

$$\begin{aligned} f'(x) &= \sum_{n=1}^{\infty} \frac{a_n}{(2n-1)!} x^{2n-1} \\ &= \frac{x}{\sqrt{1-x^2}\,(3-x^2) - 2\,(1-x^2)} \\ & \sum_{n=0}^{\infty} \frac{a_{n+1}}{(2n+1)!} x^{2n} \\ &= \frac{1}{\sqrt{1-x^2}\,(3-x^2) - 2\,(1-x^2)} \\ &= \frac{1}{(x^2-3)\sum_{n=1}^{\infty} k_n x^{2n} + 1 + x^2} \\ &= \frac{1}{1 - \frac{1}{2}x^2 + \sum_{n=2}^{\infty}(k_{n-1} - 3k_n)\,x^{2n}} \\ &= \frac{1}{\sum_{m=0}^{\infty} b_m x^{2m}} \\ k_n &= \frac{1}{2}, \frac{1}{8}, \frac{1}{16}, \frac{5}{128}, n \geq 1 \\ k_n &= \frac{(2n-2)!}{(n!)(n-1)!2^{2n-1}} \\ & k_{n-1} - 3k_n \\ &= \frac{(2n-4)!}{(n-1)!\,(n-2)!2^{2n-3}} \\ & - \frac{3\,(2n-2)!}{(n!)(n-1)!2^{2n-1}} \\ &= \frac{(2n-4)!}{(n!)(n-1)!2^{2n-1}} \times \\ & (4n\,(n-1) - 3\,(2n-2)\,(2n-3)) \\ &= \frac{-(2n-4)!2\,(n-1)\,(4n-9)}{(n!)(n-1)!2^{2n-1}} \\ &= -\frac{(2n-4)!\,(4n-9)}{(n!)(n-2)!2^{2n-2}} \end{aligned}$$

$$
\begin{aligned}
b_{0,1,2\cdots} &= 1, -\frac{1}{2}, k_{n-1} - 3k_n \\
1 &= \sum_{n=0}^{\infty} \frac{a_{n+1}}{(2n+1)!} x^{2n} \sum_{m=0}^{\infty} b_m x^{2m} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{a_{k+1} b_{n-k}}{(2k+1)!} x^{2n} \\
0 &= \frac{a_2}{3!} + a_1 b_1 \\
0 &= \sum_{k=0}^{n} \frac{a_{k+1} b_{n-k}}{(2k+1)!}
\end{aligned}
$$

Then we get the recurrent formula

$$
\begin{aligned}
a_1 &= 1 \\
b_{0,1,2} &= 1, -\frac{1}{2}, \frac{1}{8} \\
b_n &= -\frac{(2n-4)!\,(4n-9)}{(n!)\,(n-2)!2^{2n-2}}, n \geq 2, \frac{10! * 19}{7! * 5! *} \\
a_{n+1} &= -(2n+1)! \sum_{k=0}^{n-1} \frac{a_{k+1} b_{n-k}}{(2k+1)!} \\
&= \sum_{k=1}^{n-1} \frac{a_k b_{n+1-k}}{(2k-1)!} + n\,(2n+1)\,a_n \\
&= \frac{1}{2^{2n}} \sum_{k=1}^{n-1} A_{n,k} a_k + n\,(2n+1)\,a_n \\
A_{n,k} &= \frac{2^{2k}\,(2n-2k-2)!\,(4n-4k-5)\,(2n+1)!}{(2k-1)!\,(n-k+1)!\,(n-k-1)!} \\
A_{n,n-1} &= \frac{-2^{2n-2}\,(2n+1)!}{(2n-3)!2!} \\
&= -3 \times 2^{2n} C_{2n+1}^4 \\
A_{n,k+1} &= \frac{2^{2k+2}\,(2n-2k-4)!\,(4n-4k-9)\,(2n+1)!}{(2k+1)!\,(n-k)!\,(n-k-2)!} \\
\frac{A_{n,k}}{A_{n,k+1}} &= \frac{C_{2n-2k-2}^2\,(4n-4k-5)}{(4n-4k-9)} \frac{C_{2k+1}^2}{(n-k+1)\,(n-k-1)} \\
&= \frac{C_{2k+1}^2 C_{2n-2k-2}^2\,(4n-4k-5)}{(4n-4k-9)\left((n-k)^2-1\right)} \\
A_{n,k} &= \frac{C_{2k+1}^2 C_{2n-2k-2}^2\,(4n-4k-5)}{(4n-4k-9)\left((n-k)^2-1\right)} A_{n,k+1}
\end{aligned}
$$

2 Some Properties of a_n

2.1 Factors of a_n

There are some properties about a_n. It is observed that

$$
\begin{aligned}
n &\equiv 0\,(mod\ 3)\,, n \geq 2 \\
n &\equiv 0\,(mod\ 5)\,, n \geq 3 \\
a_n &\equiv 0\left(mod\ 3^{f_{n,3}}\right) \\
a_n &\equiv 0\left(mod\ 5^{f_{n,5}}\right) \\
f_{n,3} &= 0,1,1,2,4,4,7,6,6,8, \\
& \quad 9,9,10,13,13,14,17, \\
& \quad 15,17,18,19,19,21, \\
& \quad 21,24,23 \\
f_{n,5} &= 0,0,1,1,1,2,2,3,3,3, \\
& \quad 5,5,6,6,6,7,7,8,8,8\cdots
\end{aligned}
$$

2.2 Denominator of $\frac{a_n}{(2n)!}$

It is more interesting about the numerator and denominator of $\frac{a_n}{(2n)!}$

$$
\begin{aligned}
\frac{a_n}{(2n)!} &= \frac{p_n}{q_n} \\
q_n &= 2^{n_2} m, 2 \nmid m \\
n &= 2^{[\log_2^n]} + k \\
f_k &= 2n - n_2 \\
&= 1,2,2,3,2,3,3,4, \\
& \quad 2,3,3,4,3,4,4,5, \\
& \quad 2,3,3,4,3,4,4,5,\cdots
\end{aligned}
$$

There is an interesting property with the sequence f_n, one can determine f_n with a folding algorithm

$$
\begin{aligned}
f_0 &= 1 \\
f_n &= f_{n-2^{[\log_2^n]}} \\
f_{2^n+k} &= f_k, \geq 0, \\
0 &\leq k \leq 2^n - 1
\end{aligned}
$$

Or more directly, f_n is the number of 1s in the binary form of n. For instance

$$\begin{aligned} 2^m &= 1\underbrace{00\cdots0}_{m} \\ f_{2^m} &= 1 \\ 2^m-1 &= \underbrace{111\cdots1}_{m} \\ f_{2^m-1} &= m \\ f_0 &= 0 \\ f_{2n} &= f_n \\ f_{2n+1} &= f_n+1 \\ f_n &= f_{\left[\frac{n}{2}\right]} + n\ mod\ 2 \end{aligned}$$

See https://oeis.org/A000120, which is called Haming weight of n.

Finally, one can write

$$\begin{aligned} k &= n - 2^{[\log_2^n]} \\ q_n &\equiv 0 \left(mod\ 2^{2n-f_k}\right) \end{aligned}$$

Especially, when n is the power of 2, one can write

$$q_{2^m} = 2^{2^{m+1}-1} \tag{1}$$

Furthermore, when n is the multiplication of a prime and a power of 2, one can write

$$\begin{aligned} n &= 2^m p \\ q_n &= p \times 2^{2n-f_k} \end{aligned}$$

And, when n is a Mersenne prime, one can get q_n via

$$q_{2^m-1} = 2^{2^{m+1}-2(m+1)}\left(2^m-1\right)$$

where m is prime and 2^m-1 is prime.

2.3 Numerator of $\frac{a_n}{(2n)!}$

As for the numerators p_n, one can prove that all of p_n is odd since the denominators are always even. Also, there is a conjecture that there are infinitely n so that p_n is prime, some prime p_n are listed in the Appendix.

3 Summary

The Bowl's sequence was given in this note, and the recurrence formulas was given too.

Take-home message is here

(a.) The power series of arctan $\left(1-\sqrt{1-x^2}\right)$ is

$$\begin{aligned} &\arctan\left(1-\sqrt{1-x^2}\right) \\ &= \sum_{n=1}^{\infty} \frac{a_n}{(2n)!} x^{2n} \\ a_1 &= 1 \\ a_{n+1} &= \frac{1}{2^{2n}} \sum_{k=1}^{n-1} A_{n,k} a_k + n\left(2n+1\right) a_n \end{aligned}$$

(b.) A property of a_n

$$\begin{aligned} \frac{p_n}{q_n} &= \frac{a_n}{(2n)!} \\ q_{2^m} &= 2^{2^{m-1}} \end{aligned}$$

Appendix

a_n with $n \leq 20$

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 3 \\ a_3 &= 15 \\ a_4 &= 315 \\ a_5 &= 36855 \\ a_6 &= 4833675 \\ a_7 &= 711485775 \\ a_8 &= 133449190875 \\ a_9 &= 33399969978375 \\ a_{10} &= 10845524928112875 \\ a_{11} &= 4368604540935009375 \\ a_{12} &= 2121018409773134 \\ & 746875 \\ a_{13} &= 1222083076784378 \end{aligned}$$

$$\begin{aligned}
& 918484375 \\
a_{14} &= 8260130176741322 \\
& 44878796875 \\
a_{15} &= 6477241138419361 \\
& 42199672859375 \\
a_{16} &= 5831696435249193 \\
& 52829283528046875 \\
a_{17} &= 59735917746314449 \\
& 1308077497692734375 \\
a_{18} &= 69070563474827507 \\
& 73890069987317042 \\
& 96875 \\
a_{19} &= 89530877093508634 \\
& 76640969618485120 \\
& 87109375 \\
a_{20} &= 12930147565151909 \\
& 18978281765759779 \\
& 961360546875
\end{aligned}$$

p_n with $n \leq 20$

$$\begin{aligned}
p_1 &= 1 \\
p_2 &= 1 \\
p_3 &= 1 \\
p_4 &= 1 \\
p_5 &= 13 \\
p_6 &= 31 \\
p_7 &= 117 \\
p_8 &= 209 \\
p_9 &= 3077 \\
p_{10} &= 5843 \\
p_{11} &= 22415 \\
p_{12} &= 43015 \\
p_{13} &= 330457 \\
p_{14} &= 636347 \\
p_{15} &= 2458109 \\
p_{16} &= 4759409 \\
p_{17} &= 147733749 \\
p_{18} &= 287091419 \\
p_{19} &= 1117521323 \\
p_{20} &= 2178052043 \\
p_{21} &= 5667208289 \\
p_{22} &= 33216221057
\end{aligned}$$

q_n with $n \leq 20$

$$\begin{aligned}
q_1 &= 2 \\
&= 2^1 \\
q_2 &= 8 \\
&= 2^3 \\
q_3 &= 48 \\
&= 2^4 \times 3^1 \\
q_4 &= 128 \\
&= 2^7 \\
q_5 &= 1280 \\
&= 2^8 \times 5^1 \\
q_6 &= 3072 \\
&= 2^{10} \times 3^1 \\
q_7 &= 14336 \\
&= 2^{11} \times 7^1 \\
q_8 &= 32768 \\
&= 2^{15} \\
q_9 &= 589824 \\
&= 2^{16} \times 3^2 \\
q_{10} &= 1310720 \\
&= 2^{18} \times 5^1 \\
q_{11} &= 5767168 \\
&= 2^{19} \times 11^1 \\
q_{12} &= 12582912 \\
&= 2^{22} \times 3^1 \\
q_{13} &= 109051904 \\
&= 2^{23} \times 13^1 \\
q_{14} &= 234881024 \\
&= 2^{25} \times 7^1
\end{aligned}$$

$q_{15} = 1006632960$

$= 2^{26} \times 3^{1} \times 5^{1}$

$q_{16} = 2147483648$

$= 2^{31}$

$q_{17} = 73014444032$

$= 2^{32} \times 17^{1}$

$q_{18} = 154618822656$

$= 2^{34} \times 3^{2}$

$q_{19} = 652835028992$

$= 2^{35} \times 19^{1}$

$q_{20} = 1374389534720$

$= 2^{38} \times 5^{1}$

$q_{21} = 3848290697216$

$= 2^{39} \times 7^{1}$

$q_{22} = 24189255811072$

$= 2^{41} \times 11^{1}$

Prime p_n within 1000

$p_{5} = 13$

$p_{6} = 31$

$p_{10} = 5843$

$p_{15} = 2458109$

$p_{18} = 287091419$

$p_{24} = 254342741399$

$p_{35} = 3529501245305884867$

$p_{39} = 4278600287931032529 67$

$p_{83} = 90637709995720273916 8$
43972962527664171028 1
62149

$p_{104} = 89004145309737299486 3$
38995264881970166913 9
68061129264810447

$p_{109} = 17804156798361361268 5$
17077656422868387101 8
80435166581767973420 3

$p_{120} = 59303474458606810956 5$
07537756160422077630 0
62324724810592779576 3
23897

$p_{437} = 13282619876340131328 0$
58511993452510738271 1
28445221452979892008 8
06995273822141354057 5
64632345575636323268 2
09522083066120397631 8
73004111023695573776 1
42070893923950435512 7
80464355141633066366 3
99721083205353650004 5
42363591086967573041 2
88502247145242154726 6
30814189

$p_{490} = 50881833270897866310 3$
10791152762018450258 6
07351119466626979406 5
76310525380302130554 5
33161339117905017991 8
75072261988597446423 9
45338053392703373558 7
85176836785726339636 9
99298900822134447830 5
63324674831105317834 4
49317034529817642217 0
12771765569490828430 3
18057746289754048017 2
026642211891918649

$p_{552} = 2039052684120318062$
9032554889106983231
8156458396082806651
0840476779448769104
4058884861552698620
89393129452028971
4756847971993219664

2272381333662993054
3935108519650164989
0540453125429789130
04252955375193693
6274967897166378869
3431710245426744510
5743490756545708646
119920490625
5146902790276966220
8837000889825196913
98449971025842337
… … …

$$\begin{aligned} &= 1 - \cos\theta = 2\sin^2\frac{\theta}{2} \\ \int_0^\pi d\theta \cos^{2n}\theta &= 2^{-2n} C_{2n}^n \pi \end{aligned}$$

Integral of $f(x)$ in the region $[0, 1]$

$$\begin{aligned}
\int_0^1 f(x)dx &= \int_0^1 dx \arctan\left(1 - \sqrt{1 - x^2}\right) \\
&= \int_0^{\frac{\pi}{2}} d\theta \cos\theta \arctan\left(1 - \cos\theta\right) \\
&= \int_0^\pi d\theta \sum_{n=0}^{\infty} \frac{(-1)^n \cos\theta \left(1 - \cos\theta\right)^{2n+1}}{(2n+1)!} \\
&= \pi \sum_{n=0}^{\infty} \sum_{k=1}^{n+1} \frac{(-1)^{n+1} C_{2n+1}^{2k-1} C_{2k}^k}{(2n+1)! 2^{2k}} \\
&= \sum_{n=1}^{\infty} \frac{a_n}{(2n+1)!} \\
& \int_0^\pi d\theta \cos\theta \left(1 - \cos\theta\right)^{2n+1} \\
&= \int_0^\pi d\theta \cos\theta \sum_{k=0}^{2n+1} C_{2n+1}^k \left(-1\right)^k cos^k\theta \\
&= -\int_0^\pi d\theta \sum_{k=1}^{n+1} C_{2n+1}^{2k-1} cos^{2k}\theta \\
&= -\pi \sum_{k=1}^{n+1} C_{2n+1}^{2k-1} 2^{-2k} C_{2k}^k \\
x &= \sin\theta \\
\tan y &= 1 - \sqrt{1 - x^2}
\end{aligned}$$