On the Even Solutions of $\chi(n) = \chi(n+1)$

Abstract

We consider even solutions of the equation $\chi(n) = \chi(n+1)$, where χ is the alternating-sum-of-divisors function. We show that each even solution satisfies at least one of three specific congruences. We also propose an open problem regarding these solutions.

1 Introduction

In Number Theory, a well-known and extensively examined type of problem is the analysis concerning equations of the form:

$$f(n) = f(n+1),$$

where f is typically used to represent a specific arithmetic function. For example, the sequence A002961 is a list of solutions of $\sigma(n) = \sigma(n+1)$, where $\sigma(n)$ is the sum of positive divisors of n. Another commonly asked question is to analyze solutions of: $\varphi(n) = \varphi(n+1)$, where $\varphi(n)$ is the Euler totient function. The sequence A001274 is a list of solutions of this equation. In our paper, we analyze even solutions of the equation

$$\chi(n) = \chi(n+1), \tag{1}$$

where $\chi(n)$ is the alternating-sum-of-divisors function. Particularly, $\chi(n)$ is defined as follows.

Definition 1. Let the divisors of n be written as $1 = d_1 < d_2 < \ldots < d_t = n$, where $t = \tau(n)$. The *alternating sum-of-divisors function* $\chi(n)$ is defined as

$$\chi(n) = \sum_{0 \le i < t} (-1)^i d_{t-i}.$$

The function $\chi(n)$ was first introduced in 2009 by Atanassov [1]. The function $\chi(n)$ is also present in the OEIS as A071324. The solutions of Equation 1 (OEIS A333261) begins with

$$1, 5, 51, 68, 87, 116, 171, 176, 591, 2108, \ldots$$

In our paper, we prove that each even solution of Equation 1 satisfies at least one of three specific congruences.

1.1 Notation

We fix some notation. Let $\tau(n)$ be the number of divisors of n. Let $P^{-}(n)$ refer to the smallest prime factor of the integer n greater than one. Allow $p_2(n)$ to be the least factor of composite n which is strictly greater than $P^{-}(n)$. Throughout the paper, the letter k will refer to even solutions of Equation 1. Allow $\mathbb{K}$ to be the set containing all the possible values of k. Let $\omega(n)$ be the number of distinct prime factors of n.

Let $\mathcal{S} = \{n \in \mathbb{N} : n \equiv 24 \pmod{30}\} \cup \{n \in \mathbb{N} : n \equiv 20 \pmod{30}\} \cup \{n \in \mathbb{N} : n \equiv 8 \pmod{12}\}$. For any subset A of the positive integers, let $d(A)$ denote the asymptotic density of A, if it exists.

1.2 Main result and applications

The theorem presented below is the primary result of the paper.

Theorem 2. *Each k satisfies at least one of the following congruences:*

$$k \equiv 24 \pmod{30} \quad k \equiv 8 \pmod{12} \quad k \equiv 20 \pmod{30}.$$

An immediate application of Theorem 2 is in optimizing computer searches for k (even terms of A333261), since, due to Theorem 2, the search space can be reduced from all even numbers to numbers in the set $\mathcal{S}$ only. Analogous computer searches regarding equations of the form $f(n) = f(n+1)$ have received considerable attention in the mathematical literature. For example, in 1961, Makowski [5] listed the nine solutions of $\sigma(n) = \sigma(n+1)$ with n less than 10^4. A recent paper of Benito [2] discusses solutions of $\sigma(n) = \sigma(n+1)$ for $n \leq 1.5 \cdot 10^{10}$.

In 1936, Erdős [3] proved a strong result that the set containing all the solutions to $\varphi(n) = \varphi(n+1)$ has asymptotic density zero. A weaker yet similar result concerning $d(\mathbb{K})$ (if it exists) can be immediately derived through Theorem 2. Particularly, since $\mathbb{K}$ is a strict subset of $\mathcal{S}$, we obtain the result

Proposition 3. *If $d(\mathbb{K})$ exists, then $d(\mathbb{K}) \leq d(\mathcal{S}) < 0.15$.*

We leave it as an exercise for the reader to prove that $d(\mathcal{S})$ exists and is strictly less than 0.15.

1.3 Proof strategy for Theorem 2

For proving Theorem 2, we first prove some preliminary bounds for $\chi(n)$, which will be subsequently used. Through these bounds, we restrict values of $P^{-}(k+1)$ and $p_2(k)$. The restrictions on the values of $P^{-}(k+1)$ and $p_2(k)$ enable us to obtain the result that there can be only six distinct combinations of $P^{-}(k+1)$ and $p_2(k)$ values. Further analysis allows us to eliminate three combinations, leaving only three permissible combinations of $P^{-}(k+1)$ and $p_2(k)$. Through these permissible combinations, we obtain factors of k and $k+1$ for each permissible combination case, which we then translate into corresponding congruences.

1.4 Organization of the paper

The paper is structured as follows. Section 2 presents some preliminary bounds regarding $\chi(n)$. Section 3 will discuss restrictions on $P^-(k+1)$ and $p_2(k)$ values. Section 4 showcases the elimination of some combinations of $P^-(k+1)$ and $p_2(k)$, and Section 5 will put forward the final permissible combinations of $P^-(k+1)$ and $p_2(k)$. Finally, Section 6 will present a proof for Theorem 2. Section 7 will present an open problem regarding solutions of Equation 1. In the end of the paper, an appendix will also be provided, containing a table with the values: k, $\chi(k)$, the corresponding congruence, $\tau(k)$, $\sigma(k)$, and $\omega(k)$. The values presented in Table 2 are calculated based on the known terms of the sequence A333261.

2 Preliminary theorems regarding bounds for $\chi(n)$

This section will establish fundamental bounds related to $\chi(n)$.

Theorem 4. *For n greater than one, we have*

$$\frac{\chi(n)}{n} \geq 1 - \frac{1}{P^-(n)}.$$

Proof. For n greater than one, we have $d_{t-1} = n/P^-(n)$. Hence, we note that

$$\chi(n) = \sum_{0 \leq i < t} (-1)^i d_{t-i} = n - \frac{n}{P^-(n)} + \sum_{2 \leq i < t} (-1)^i d_{t-i}.$$

But if we group the elements of the sum: $\sum_{2 \leq i < t} (-1)^i d_{t-i}$ in pairs, we see that the first number of the pair is always greater than the second number, so:

$$\chi(n) \geq n - \frac{n}{P^-(n)} = n\left(1 - \frac{1}{P^-(n)}\right). \tag{2}$$

Inequality 2 allows us to conclude that the inequality:

$$\frac{\chi(n)}{n} \geq 1 - \frac{1}{P^-(n)}$$

holds for $n > 1$. □

Theorem 4 provides a lower bound for $\chi(n)/n$ when n is odd.

Corollary 5. *For odd n, we have $\chi(n)/n \geq 2/3$.*

Proof. We note that $\chi(1)/1 = 1$, which is greater than $2/3$. For odd n greater than one, $P^-(n)$ is at least three. Hence, by Theorem 4, the following can be concluded:

$$\frac{\chi(n)}{n} \geq 1 - \frac{1}{P^-(n)} \geq 1 - \frac{1}{3} = \frac{2}{3}.$$

□

We now demonstrate an upper bound for $\chi(n)/n$.

Theorem 6. *For even n greater than two, we have*

$$\frac{\chi(n)}{n} \leq \frac{1}{2} + \frac{1}{p_2(n)}.$$

Proof. For even n greater than two, we note that $d_{t-1} = n/2$ and $d_{t-2} = n/p_2(n)$. As a result:

$$\chi(n) = \sum_{0\leq i<t} (-1)^i d_{t-i} = n - \frac{n}{2} + \frac{n}{p_2(n)} + \sum_{3\leq i<t} (-1)^i d_{t-i}.$$

If we group the elements of the sum $\sum_{3\leq i<t}(-1)^i d_{t-i}$ in pairs, we see that the first number of the pair is always greater than the second number, hence we conclude that

$$\chi(n) \leq \frac{n}{2} + \frac{n}{p_2(n)},$$

which implies

$$\frac{\chi(n)}{n} \leq \frac{1}{2} + \frac{1}{p_2(n)},$$

as desired. □

Theorem 6 provides an upper bound for $\chi(n)/n$ for even n.

Corollary 7. *For even n, we have $\chi(n)/n \leq 5/6$.*

Proof. We note that $\chi(2)/2 = 1/2$, which is less than $5/6$. Since for even n greater than two, $p_2(n)$ is at least three, therefore we conclude that

$$\frac{\chi(n)}{n} \leq \frac{1}{2} + \frac{1}{p_2(n)} \leq \frac{1}{2} + \frac{1}{3} = \frac{5}{6}.$$

□

3 Restricting $P^-(k+1)$ and $p_2(k)$ values

In this section, we outline important theorems that restrict the values of $P^-(k+1)$ and $p_2(k)$, which will be crucial for establishing the required permissible combinations of $P^-(k+1)$ and $p_2(k)$. We begin by noting a simple relation.

Lemma 8. *The following inequality holds for each k:*

$$\frac{5}{6} \geq \frac{\chi(k)}{k} > \frac{\chi(k+1)}{k+1} \geq \frac{2}{3}.$$

Proof. The inequality:

$$\frac{5}{6} \geq \frac{\chi(k)}{k}$$

is a direct consequence of Corollary 7, as k is even. Since $\chi(k) = \chi(k+1)$, we note that

$$\frac{\chi(k)}{k} > \frac{\chi(k+1)}{k+1}.$$

Finally, by Corollary 5, we obtain

$$\frac{\chi(k+1)}{k+1} \geq \frac{2}{3}.$$

Combining these inequalities leads us to the desired result. □

The direct implication of Lemma 8 is as follows.

Proposition 9. *For each k, the ratio $\chi(k)/k$ is greater than $2/3$ and the ratio $\chi(k+1)/(k+1)$ is less than $5/6$.*

We now move on to the first key theorem of this section, which restricts the values of $p_2(k)$.

Theorem 10. *For each k, we have*

$$p_2(k) \in \{3, 4, 5\}.$$

Proof. By Theorem 6 and Proposition 9, we have

$$\frac{1}{2} + \frac{1}{p_2(k)} \geq \frac{\chi(k)}{k} > \frac{2}{3},$$

which implies that:

$$\frac{1}{2} + \frac{1}{p_2(k)} > \frac{2}{3}. \tag{3}$$

Inequality 3 implies that:

$$6 > p_2(k).$$

As k is even, hence, $6 > p_2(k) > 2$, which proves that $p_2(k)$ can be equal to three, four, or five only. □

We now discuss the second theorem of this section, which places restrictions on the possible values of $P^-(k+1)$.

Theorem 11. *For each k, we have*

$$P^-(k+1) \in \{3, 5\}.$$

Proof. By Theorem 4 and Proposition 9, we get

$$\frac{5}{6} > \frac{\chi(k+1)}{k+1} \geq 1 - \frac{1}{P^-(k+1)},$$

which implies that

$$\frac{5}{6} > 1 - \frac{1}{P^-(k+1)}. \tag{4}$$

As a result of Inequality 4, we obtain that $P^-(k+1)$ is less than six. As $k+1$ is odd, we conclude that $6 > P^-(k+1) > 2$. Hence, $P^-(k+1)$ can only be three or five. □

4 Eliminating combinations of $P^-(k+1)$ and $p_2(k)$

Theorem 10 and Theorem 11 restrict the values of $p_2(k)$ and $P^-(k+1)$ to the sets $\{3, 4, 5\}$ and $\{3, 5\}$ respectively. Hence, there can only be six possible combinations of $P^-(k+1)$ and $p_2(k)$. In this section, we eliminate some combinations of $P^-(k+1)$ and $p_2(k)$. We begin by eliminating the most immediate combinations.

Proposition 12. *For each k, the following conditions hold.*

- *If $p_2(k) = 3$, then, $P^-(k+1) \neq 3$.*
- *If $p_2(k) = 5$, then, $P^-(k+1) \neq 5$.*

Proof. The proof is quite simple. If $p_2(k) = 3$, then, 3 is a factor of k, hence $k+1$ cannot have 3 as a factor. Hence, $P^-(k+1) \neq 3$. A similar argument can be applied to the case of $p_2(k) = 5$. □

We will now eliminate one additional combination.

Proposition 13. *For each k, if $p_2(k) = 4$, then, $P^-(k+1) \neq 5$.*

Proof. Combining Lemma 8, Theorem 4 and Theorem 6, we obtain that

$$\frac{1}{2} + \frac{1}{p_2(k)} \geq \frac{\chi(k)}{k} > \frac{\chi(k+1)}{k+1} \geq 1 - \frac{1}{P^-(k+1)}.$$

As a result, the inequality:

$$\frac{1}{2} + \frac{1}{p_2(k)} > 1 - \frac{1}{P^-(k+1)}, \tag{5}$$

holds for each permissible combination of $p_2(k)$ and $P^-(k+1)$. It is easy to check that the combination of $p_2(k) = 4$ and $P^-(k+1) = 5$ does not satisfy Inequality 5. It should be noted that, apart from the combination of $p_2(k) = 4$ and $P^-(k+1) = 5$, the combination of $p_2(k) = 5$ and $P^-(k+1) = 5$ is also ruled out by Inequality 5. All other combinations satisfied Inequality 5. □

5 Permissible combinations of $P^{-}(k+1)$ and $p_2(k)$

In this section, we apply Theorem 10, Theorem 11, Proposition 12 and Proposition 13 to obtain all the permissible combinations of $p_2(k)$ and $P^{-}(k+1)$. We present the following theorem as our principal result.

Theorem 14. *For each k, all the possible permissible combinations of $p_2(k)$ and $P^{-}(k+1)$ are as follows:*

- *If $p_2(k) = 3$, then $P^{-}(k+1) = 5$.*
- *If $p_2(k) = 4$, then $P^{-}(k+1) = 3$.*
- *If $p_2(k) = 5$, then $P^{-}(k+1) = 3$.*

Proof. We note that through Theorem 10 and Theorem 11, one obtains the following possible combinations of $p_2(k)$ and $P^{-}(k+1)$:

Index	$p_2(k)$	$P^{-}(k+1)$
1	3	3
2	4	3
3	5	3
4	3	5
5	4	5
6	5	5

Table 1: All the possible combinations of $p_2(k)$ with corresponding $P^{-}(k+1)$, according to Theorem 10 and Theorem 11.

Through Proposition 12, the combinations with index numbers 1 and 6 in Table 1 are eliminated. Similarly, by applying Proposition 13, we can eliminate the combination with the index number 5. Therefore, we are left with an exhaustive list of all the permissible combinations. □

6 Proof of Theorem 2

We present the proof of Theorem 2.

Proof. Through Theorem 14, we can obtain factors of k and $k+1$ for each permissible combination case. We convert these factors into congruence relations for k and $k+1$, and then we apply the Chinese remainder theorem to obtain the corresponding congruence for k. Since every possible k satisfies at least one of the permissible combinations, it is guaranteed that each k will satisfy at least one congruence, since each permissible combination has a corresponding congruence. In the subsequent section, we will analyze each permissible combination case in detail.

- (Case 1): If $p_2(k) = 3$, then $P^-(k+1) = 5$. Since k is even, this combination implies that 6 is a factor of k, and 5 is a factor of $k+1$. Therefore, $k \equiv 0 \pmod{6}$, and $k+1 \equiv 0 \pmod{5}$. As a result, we conclude that $k \equiv 24 \pmod{30}$.

- (Case 2): If $p_2(k) = 4$, then $P^-(k+1) = 3$. This combination implies that 4 is a factor of k and 3 is a factor of $k+1$. Hence $k \equiv 0 \pmod{4}$ and $k+1 \equiv 0 \pmod{3}$. After solving, we obtain that $k \equiv 8 \pmod{12}$.

- (Case 3): If $p_2(k) = 5$, then $P^-(k+1) = 3$. This combination implies that 10 is a factor of k and 3 is a factor of $k+1$. Therefore, we obtain that, $k \equiv 0 \pmod{10}$ and $k+1 \equiv 0 \pmod{3}$. As a result, we conclude that $k \equiv 20 \pmod{30}$.

These arguments conclude the proof of Theorem 2. □

7 An open problem regarding solutions of Equation 1

In this section, we propose an open problem regarding the solutions of Equation 1.

Conjecture 15. There are infinitely many solutions to Equation 1.

In the mathematical literature, several conjectures analogous to Conjecture 15 have been made. One of these conjectures is the Erdős-Sierpiński problem. First put forward by Erdős [4] and later re-asked by Sierpiński [6, p. 177], this conjecture states that there are infinitely many solutions to the equation $\sigma(n) = \sigma(n+1)$. Another such problem was also asked by Tóth [7], who asked whether there are infinitely many solutions of the equation $\beta(n) = \beta(n+1)$, where $\beta(n)$ (OEIS A206369) is another function belonging in the family of alternating-sum-of-divisors functions. The solutions of $\beta(n) = \beta(n+1)$ is the sequence A206368.

8 Appendix

We present Table 2 consisting of the values: k, $\chi(k)$, the corresponding congruence, $\tau(k)$, $\sigma(k)$, and $\omega(k)$ for k up to 6818712836.

Index	k	$\chi(k)$	**Congruence**	$\tau(k)$	$\sigma(k)$	$\omega(k)$
1	68	48	$k \equiv 8 \pmod{12}$	6	126	2
2	116	84	$k \equiv 8 \pmod{12}$	6	210	2
3	176	120	$k \equiv 8 \pmod{12}$	10	372	2
4	2108	1480	$k \equiv 8 \pmod{12}$	12	4032	3
5	9308	6480	$k \equiv 8 \pmod{12}$	12	17640	3
6	18548	13908	$k \equiv 8 \pmod{12}$	6	32466	2
7	37928	25920	$k \equiv 8 \pmod{12}$	16	77760	3
8	180548	135408	$k \equiv 8 \pmod{12}$	6	315966	2
9	192428	142560	$k \equiv 8 \pmod{12}$	12	341880	3
10	200996	149688	$k \equiv 8 \pmod{12}$	12	355740	3
11	3960896	2646960	$k \equiv 8 \pmod{12}$	28	7924800	3
12	8198156	6048000	$k \equiv 8 \pmod{12}$	12	14582400	3
13	9670748	7153920	$k \equiv 8 \pmod{12}$	12	17156160	3
14	11892512	7938000	$k \equiv 8 \pmod{12}$	24	23814000	3
15	16585748	12402720	$k \equiv 8 \pmod{12}$	12	29115072	3
16	25367396	18900000	$k \equiv 8 \pmod{12}$	12	44688000	3
17	25643012	18823200	$k \equiv 8 \pmod{12}$	12	45830400	3
18	29768312	20487168	$k \equiv 8 \pmod{12}$	32	64696320	4
19	61735352	42484608	$k \equiv 8 \pmod{12}$	32	134161920	4
20	68571248	46949760	$k \equiv 8 \pmod{12}$	20	133415568	3
21	101346368	67647600	$k \equiv 8 \pmod{12}$	28	202460352	3
22	102132290	68124672	$k \equiv 20 \pmod{30}$	32	204374016	5
23	114246470	76204800	$k \equiv 20 \pmod{30}$	32	228614400	5
24	166123268	124592448	$k \equiv 8 \pmod{12}$	6	290715726	2
25	228081452	162086400	$k \equiv 8 \pmod{12}$	24	425476800	4
26	250391552	166927704	$k \equiv 8 \pmod{12}$	44	502579440	3
27	514531676	375701760	$k \equiv 8 \pmod{12}$	24	935079936	4
28	804078968	555024960	$k \equiv 8 \pmod{12}$	64	1930521600	5
29	1010223896	691891200	$k \equiv 8 \pmod{12}$	32	2075673600	4
30	1153706948	857623200	$k \equiv 8 \pmod{12}$	12	2036855100	3
31	1338817292	953557920	$k \equiv 8 \pmod{12}$	24	2472187200	4
32	2005484096	1354872960	$k \equiv 8 \pmod{12}$	112	5133075840	5
33	2676172592	1839166560	$k \equiv 8 \pmod{12}$	20	5187219336	3
34	3386945432	2288563200	$k \equiv 8 \pmod{12}$	32	6865689600	4
35	3840293552	2596920480	$k \equiv 8 \pmod{12}$	20	7562547216	3
36	6616978928	4680178688	$k \equiv 8 \pmod{12}$	40	15288926208	4
37	6818712836	4800660480	$k \equiv 8 \pmod{12}$	24	13068464640	4

Table 2: Table of k, $\chi(k)$, the corresponding congruence, $\tau(k)$, $\sigma(k)$, and $\omega(k)$.

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