Part I Proof of the formula

Let $S(N)$ be the sum of digits of N in base b. Suppose $N, N+1, \dots, N+m-1$ satisfy: $n \nmid S(N+i)$, $i = 0,1,\dots,m-1$. We intend to find the maximum value of m.

Write $n = (b-1)s + t, 1 \le t \le b-1$. For any $M \in \mathbb{N}$, $0 \le d \le b-t$, we have:

$$S(b^{s+1}M + b^s d) = S(M) + d, S(b^{s+1}M + b^s(d+t) - 1) = S(M) + d + t - 1 + (b-1)s.$$

That is, the sums of digits of the numbers in the range $[b^{s+1}M + b^s d, b^{s+1}M + b^s(d+t) - 1]$ are n consecutive numbers, so there must be a multiple of n.

Now there are two cases:

A. $b^{s+1} \nmid N+i$, $i = 1,2,\dots,m-1$.

Then there exists some M such that $[N, N+m-1] \subseteq [b^{s+1}M, b^{s+1}(M+1) - 1]$. Note that for any $0 \le d \le b-t$ we have

$$[b^{s+1}M + b^s d, b^{s+1}M + b^s(d+t) - 1] \nsubseteq [N, N+m-1].$$

If $m \ge b^s(t+1) - 1$, write $N = b^{s+1}M + b^s d_0 + r$, $0 \le d_0 \le b-t-1$ (or $0 \le d_0 \le b-t$ if $s=0$), $0 \le r \le b^s - 1$ (if $s = 0$, then $r = 0$).

(a) If $r = 0$, then $[b^{s+1}M + b^s d_0, b^{s+1}M + b^s(d_0+t) - 1] \subseteq [N, N+m-1]$, a contradiction;

(b) If $1 \le r_0 \le b^s - 1$, then $[b^{s+1}M_0 + b^s(d_0+1), b^{s+1}M_0 + b^s(d_0+1+t) - 1] \subseteq [N, N+m-1]$, a contradiction. So

$$m \le b^s(t+1) - 2. \tag{1}$$

B. $N + i_0 = b^{s+1}M$ for some $1 \le i_0 \le m-1$. For convenience, write $g = \gcd(t, b-1)$.

Then $[N, N+m-1] \subseteq [b^{s+1}M - b^s t + 1, b^{s+1}M + b^s t - 2]$. Note that for $0 \le i \le b^{s+1} - 1$ we have

$$S(b^{s+1}M - 1 - i) = S(b^{s+1}M - 1) - S(i), S(b^{s+1}M + i) = S(b^{s+1}M) + S(i).$$

Let r_1, r_2 be the remainder of $S(b^{s+1}M - 1)$, $S(b^{s+1}M)$ modulo n respectively. By definition, $r_2 \ne 0$. $N, N+1, \dots, N+m-1$ should satisfy: $N-1$ is the largest number no greater than $b^{s+1}M - 1$ whose digits sum to $S(b^{s+1}M - 1) - r_1$; $N+m$ is the smallest number no less than $b^{s+1}M$ whose digits sum to $S(b^{s+1}M) + n - r_2$. Define

$$f(N) = \min\{N' : S(N') = N\}.$$

It is not hard to see

$$N - 1 = b^{s+1}M - 1 - f(r_1), N + m = b^{s+1}M + f(n - r_2).$$

And $m = f(r_1) + f(n - r_2)$. It is obvious that $f(N)$ is strictly increasing.

Let r_1', r_2' be the remainder of r_1, $n - r_2$ modulo g respectively. Then we have

$$\begin{aligned} f(r_1) + f(n - r_2) &\le f(n - g + r_1') + f(n - g + r_2') \\ &= b^s(t - g + r_1' + 1) - 1 + b^s(t - g + r_2' + 1) - 1, \end{aligned}$$

It is easy to know $r_2 - r_1 \equiv 1 \pmod{g}$. This gives $r_1' + r_2' = g - 1$. So we have

$$m \le b^s(2t - \gcd(t, b-1) + 1) - 2. \tag{2}$$

Compare (1) and (2), we have $m_{\max} = b^s(2t - \gcd(t, b-1) + 1) - 2$ (see the examples in Part II).

Part II The examples N_0

Now we determine the numbers N_0 such that

$$n \nmid S(N_0 + i), \ i = 0,1,\dots,m_{\max} - 1. \tag{3}$$

Again, for convenience, write $g = \gcd(t, b-1)$. Let's suppose that $n \nmid b-1$, then $n \nmid g$, and $t|b-1$ implies $s \geq 1$.

A. $N_0 + i_0$ is a multiple of b^{s+1} for some $1 \leq i_0 \leq m_{\max} - 1$.

Suppose that it is $b^u M_0$, $b \nmid M_0$, $u \geq s+1$. Let r_1, r_2 be the remainder of $S(b^u M_0 - 1)$, $S(b^u M_0)$ modulo n respectively. Recall that $N_0 = b^u M_0 - f(r_1)$. (3) is valid (i.e., $f(r_1) + f(n - r_2)$ is equal to $m_{\max}$) if and only if the values of r_1 and r_2 are:

$$(r_1, r_2) = (n-g, 1), (n-g+1, 2), \dots, (n-1, g). \tag{4}$$

$b \nmid M_0$ implies that $S(b^u M_0 - 1) = S(M_0 - 1) + (b-1)u = S(M_0) - 1 + (b-1)u$. When (4) holds, we have

$$S(M_0) - 1 + (b-1)u \equiv S(M_0) + n - g - 1 \pmod{n},$$
$$(b-1)u \equiv -g \pmod{n}. \tag{5}$$

Since $r_1 \geq n - g \geq (b-1)s$, $f(r_1) = b^s(r_1 - (b-1)s + 1) - 1 = b^s(n - g - 1 + r_2 - (b-1)s + 1) - 1 = b^s(t - g + r_2) - 1$. As a result, the necessary condition for N_0 is:

$$N_0 = b^u M_0 - b^s(t - \gcd(t, b-1) + r_2) + 1, \tag{6}$$

Where u is a nonnegative solution to (5) not equal to s, $b \nmid M_0$, $S(M_0) \equiv r_2 \pmod{n}$ and $1 \leq r_2 \leq g$.

On the other hand, (5) has no solutions in the range $[0, s-1]$. If u is a nonnegative solution to (5) not equal to s, then $u \geq s+1$, so $b^u M_0$ is indeed a multiple of b^{s+1}. It is easy to see that (4) can be derived from (6), so (6) is also sufficient for (3).

B. $b^{s+1} \nmid N_0 + i$, $i = 1, 2, \dots, m_{\max} - 1$.

This would only happen when $t|b-1$, where $m_{\max} = b^s(t+1) - 2$. Suppose that $[N_0, N_0 + m_{\max} - 1] \subseteq [b^{s+1} M_0, b^{s+1}(M_0 + 1) - 1]$. Recall that for any $0 \leq d \leq b - t$ we have

$$[b^{s+1} M_0 + b^s d, b^{s+1} M_0 + b^s(d+t) - 1] \nsubseteq [N_0, N_0 + m_{\max} - 1].$$

Write $N_0 = b^{s+1} M_0 + b^s d_0 + r_0$, $0 \leq d_0 \leq b - t - 1$, $0 \leq r_0 \leq b^s - 1$.

(a) If $r_0 = 0$, then $[b^{s+1} M_0 + b^s d_0, b^{s+1} M_0 + b^s(d_0 + t) - 1] \subseteq [N_0, N_0 + m_{\max} - 1]$, a contradiction;

(b) If $2 \leq r_0 \leq b^s - 1$, then $[b^{s+1} M_0 + b^s(d_0 + 1), b^{s+1} M_0 + b^s(d_0 + 1 + t) - 1] \subseteq [N_0, N_0 + m_{\max} - 1]$, a contradiction;

(c) If $r_0 = 1$, then the sums of digits of the numbers in the range $[N_0, N_0 + m_{\max} - 1]$ are from $S(b^{s+1} M_0 + b^s d_0 + 1) = S(M_0) + d_0 + 1$ to $S(b^{s+1} M_0 + b^s(d_0 + t) - 1) = S(b^{s+1} M_0 + b^s(d_0 + t + 1) - 2) = S(M_0) + d_0 + t + (b-1)s - 1 = S(M_0) + d_0 + n - 1$. So (3) is valid if and only if

$$N_0 = b^{s+1} M_0 + b^s d_0 + 1,$$

Where $S(M_0) \equiv -d_0 \pmod{n}$, $0 \leq d_0 \leq b - t - 1$.

(If $n|b-1$, then (3) holds if and only if $N_0 \equiv 1 \pmod{n}$. The results from either case A or case B are not suitable in this case.)

Part III The smallest example

Write

$$N_{0\min} = b^{u_0} - b^s(t - \gcd(t, b-1) + 1) + 1,$$

Where u_0 is the smallest nonnegative solution to (5). We will first show that (3) is valid if N_0 is replaced by $N_{0\min}$. This is because: if $t|b-1$, then $u_0 = s, N_{0\min} = 1$, this is the case B in Part II where $M_0 = 0$, $d_0 = 0$ when $n \nmid b-1$; if $t \nmid b-1$, then $u_0 \geq s+1$, this is the case A in

Part II where $u = u_0$, $M_0 = 1$, $r_2 = 1$.

We will show that $N_{0\min}$ is the smallest possible value of N_0 in (3). The case $t | b - 1$ is trivial. Now suppose $t \nmid b - 1$, then we have (6). Since $S(M_0) \equiv r_2 \pmod{n}$ and $r_2 \leq g \leq b - 1$, $M_0 \geq f(r_2) = r_2$, so $N_0 = b^u M_0 - b^s(t - \gcd(t, n) + r_2) + 1 \geq b^{u_0} r_2 - b^s(t - \gcd(t, n) + r_2) + 1$. As a result, we have

$$N_0 - N_{0\min} \geq (b^{u_0} - b^s)(r_2 - 1) \geq 0,$$

Which is the desired result.