


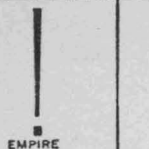
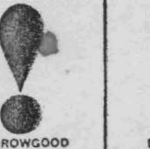











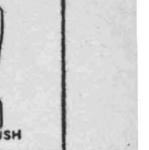

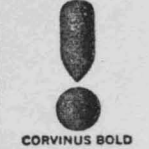






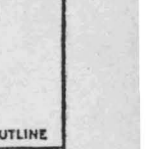



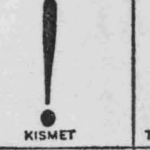

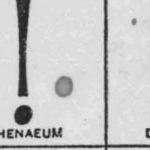
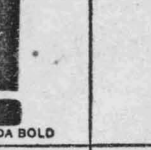

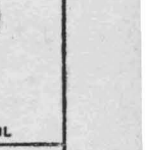




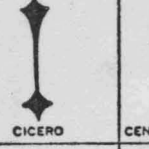
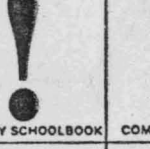
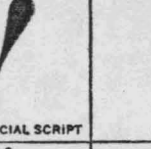





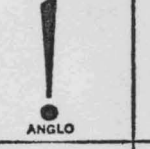




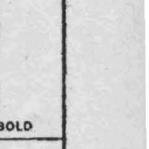





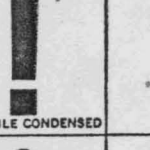

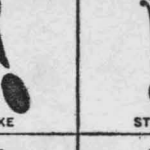
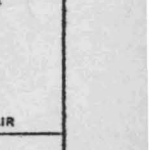



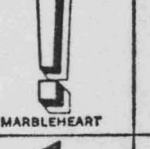

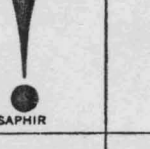
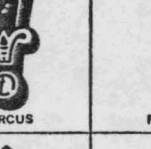
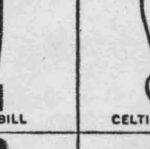
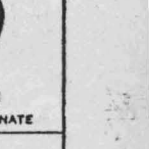


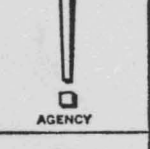
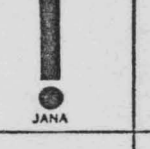
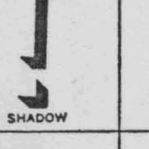
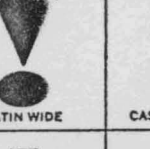
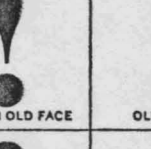
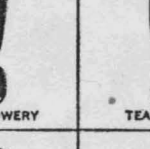
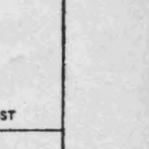



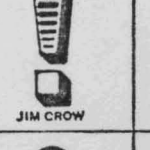
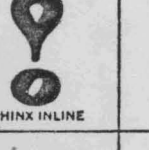
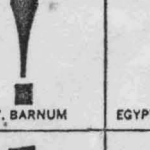
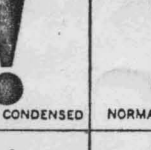
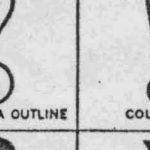
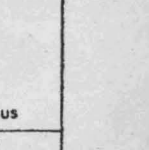





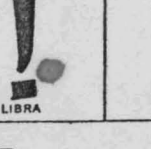
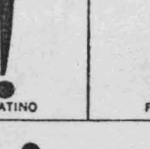
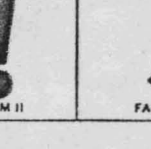
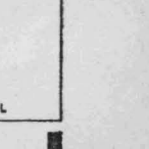
 WINDSOR	 CHELTENHAM BOLD	 GRAPHIQUE	 BROADWAY	 EMPIRE	 THOROWGOOD	 DELPHINIAN	 ULTRA BODONI ITALIC	 UMBRA
 WINDSOR	 PRISMA	 PISTILLI ROMAN	 THUNDERBIRD	 COOPER BLACK ITALIC	 GRECO ADORNADO	 BETON EXTRA BOLD	 NEULAND LICHTE	 GOLD RUSH
 FONTANESI	 CORVINUS BOLD	 CLEARFACE BOLD	 CLARENDON BOLD	 BETON OPEN	 METROPOLIS BOLD	 BULMER ITALIC	 STYMIE EXTRA BOLD	 WINDSOR OUTLINE
 CRAW MODERN	 BANK NOTE ITALIC	 NORMANDE ITALIC	 KISMET	 THOROWGOOD ITALIC	 ATHENAEUM	 DAVIDA BOLD	 ROYAL	 STENCIL
 OLYMPIC	 CARTOON BOLD	 CHELTENHAM OPEN	 PEIGNOT BOLD	 CICERO	 CENTURY SCHOOLBOOK	 COMMERCIAL SCRIPT	 AMELIA	 ALFERETA
 DEUTCH BLACK	 VISA	 COMSTOCK	 ANGLO	 FUTURA BLACK	 BOOKMAN ITALIC	 MANDARIN	 GALLIA	 HESS NEOBOLD
 COCHIN OLD STYLE	 BANCO	 CLOISTER BLACK	 ARBORET	 BODONI BOLD ITALIC	 EUROSTILE CONDENSED	 CHISEL	 SMOKE	 ST. CLAIR
 SANS SERIF SHADED	 AESTHETIC	 BRITTANIC	 MARBLEHEART	 PROFIL	 SAPHIR	 CIRCUS	 PLAYBILL	 CELTIC ORNATE
 KARNAC	 GIORGIO	 AGENCY	 JAHA	 SHADOW	 LATIN WIDE	 CASLON OLD FACE	 OLD BOWERY	 TEA CHEST
 SCOTFORD UNCIAL	 FLIRT	 EGYPTIAN OPEN	 JIM CROW	 SPHINX INLINE	 P. T. BARNUM	 EGYPTIAN CONDENSED	 NORMANDIA OUTLINE	 COLUMBUS
 TRUMP GRAVURE	 THORNE SHADED	 BASKERVILLE BOLD	 COOPER BLACK	 BAKER SIGNET	 LIBRA	 PALATINO	 FORUM II	 FANTAIL

(ZEROTH) FACTORIAL

Product definitions:

$$z! = z!_0 = e^{-\gamma z} \prod_{n \geq 1} \frac{e^{z/n}}{1+z/n}, \quad (0.\text{Prod.1})$$

$$= \prod_{n \geq 1} \frac{(n+1)^z}{(n+z)n^{z-1}} = \prod_{n \geq 1} \frac{(1+1/n)^z}{1+z/n}. \quad (0.\text{Prod.2})$$

Recurrence:

$$\frac{z!}{(z-1)!} = z. \quad (0.\text{Rec})$$

Pi:

$$\pi = \pi_0 = (-1/2)!^2 = 3.14159\ 26535\ 89793\ \dots \quad (0.\text{Pi})$$

Replication formula:

$$\prod_{i=0}^{n-1} (z - i/n)! = \frac{(nz)!(2\pi)^{\frac{n-1}{2}}}{n^{nz+1/2}}. \quad (0.\text{Rep})$$

Reflection formula:

$$z!(-z)! = \frac{\pi z}{\sin \pi z}. \quad (0.\text{Ref})$$

Asymptoty:

$$\begin{aligned} z! &= \sqrt{2\pi z} z^z e^{-z + \frac{B_2}{1 \cdot 2} z^{-1} + \frac{B_4}{3 \cdot 4} z^{-3} + \frac{B_6}{5 \cdot 6} z^{-5} + \dots} \\ &= \sqrt{2\pi z} z^z e^{-z + \frac{1}{12} z^{-1} - \frac{1}{360} z^{-3} + \frac{1}{1260} z^{-5} - \dots} \\ &= \sqrt{2\pi z} \left(\frac{z}{e}\right)^z \left(1 + \frac{1}{12z} + \frac{1}{288z^2} - \frac{139}{51840z^3} - O(z^{-4})\right) \quad (0.\text{Asym}) \\ &= \sqrt{2\pi z} \left(\frac{z}{e}\right)^z \prod_{n \geq z} \frac{(1+1/n)^{n+1/2}}{e}. \end{aligned}$$

Product of the general rational function: if $-a_1, -a_2, \dots, -a_k$ are the poles and roots and p_1, p_2, \dots, p_k are the respective multiplicities (counting poles negatively), then, since (O.Prod.2) is equivalent to

$$\prod_{n \geq 1} \frac{(n+a_1)^{p_1} (n+a_2)^{p_2} \dots (n+a_k)^{p_k}}{n^{p_1+p_2+\dots+p_k}} \left(\frac{n}{n+1} \right)^{a_1 p_1 + a_2 p_2 + \dots + a_k p_k} = \frac{1}{a_1!^{p_1} a_2!^{p_2} \dots a_k!^{p_k}},$$

(O.Genp)

the desired product converges, and to this limit, iff $\sum p_i = \sum a_i p_i = 0$.

Faster convergence: (0.Prod.1) is the special case $k = 1, m \rightarrow \infty$, of

$$\prod_{n=1}^m \frac{e^{\frac{z}{n} - \frac{1}{2}(\frac{z}{n})^2 + \frac{1}{3}(\frac{z}{n})^3 - \dots - \frac{1}{k}(\frac{z}{n})^k}}{1+z/n} = z! \left(1 - \frac{(-z)^{k+1}}{k(k+1)m^k} + O\left(\frac{z^{k+2}}{k^2 m^{k+1}}\right) \right), \quad (0.Fast.1)$$

where the exponent in the multiplicand is just k terms of $\log 1+z/n$.

Similarly, (0.Prod.2) is the special case $k = 0, m \rightarrow \infty$, of

$$\frac{\prod_{n=1}^m \frac{(n+1)^{\frac{z+k}{k+1}}}{n^{\frac{z-1}{k+1}}} \prod_{i=0}^k (n+z+i) \frac{(-1)^{i+1} (z-1)z \dots (z+i-2)(z+i+1) \dots (z+k)}{i!(k-i)!(k+1)}}{\prod_{i=1}^k ((z+1)(z+2) \dots (z+i)) \frac{(-1)^i (z-1)z \dots (z+i-2)(z+i+1) \dots (z+k)}{i!(k-i)!(k+1)}} = z! \left(1 - \frac{(z-1)z \dots (z+k)}{(k+1)^2(k+2)m^{k+1}} + O\left(\frac{1}{k^3 m^{k+2}}\right) \right). \quad (0.Fast.2)$$

Taylor series: sending $k \rightarrow \infty$ in (0.Fast.1),

$$z! = e^{-\gamma z + \frac{\zeta(2)}{2} z^2 - \frac{\zeta(3)}{3} z^3 + \frac{\zeta(4)}{4} z^4 - \dots} \quad (0.Tay.e)$$

$$= 1 - \gamma z + \frac{6\gamma^2 + \pi^2}{12} z^2 - \frac{2\gamma^3 + \gamma\pi^2 + 4\zeta(3)}{12} z^3 + \dots \quad (0.Tay)$$

$$\frac{(z-1/2)!}{\sqrt{\pi}} = 4^{-z} e^{-\gamma z + \frac{3\zeta(2)}{2} z^2 - \frac{7\zeta(3)}{3} z^3 + \frac{15\zeta(4)}{4} z^4 - \dots} \quad (0.Tay.e.5)$$

$$= 1 - gz + \frac{2g^2 + \pi^2}{4} z^2 - \frac{2g^3 + 3\pi^2 g + 28\zeta(3)}{12} z^3 + \dots, \quad (0.Tay.5)$$

where $g = \gamma + 2 \ln 2$.

FIRST FACTORIAL

Definition:

$$z! = \frac{e^{\int_0^z \ln t! dt + \frac{z(z+1)}{2}}}{(2\pi)^{z/2}} \quad (1.Def)$$

Product definitions:

$$z! = \left(\frac{e^{\frac{z+\gamma_{z+1}}{2}} z!}{\sqrt{2\pi}} \right)^z \prod_{n \geq 1} \frac{e^{-z^2/2n}}{(1+z/n)^n} \quad (1.Prod.0)$$

$$= \frac{e^{\frac{(z-\gamma_{z+1})z}{2}}}{(2\pi)^{z/2}} \prod_{n \geq 1} \frac{e^{z+z^2/2n}}{(1+z/n)^{n+z}} \quad (1.Prod.1)$$

$$= e^{\frac{(z-1)z}{2}} \prod_{n \geq 1} \frac{(1+1/n)^{(n+\frac{z+1}{2})z}}{(1+z/n)^{n+z}} \quad (1.Prod.2)$$

$$= \left(\frac{e^{z+1} z!}{2\pi} \right)^{z/2} \prod_{n \geq 1} \frac{e^z}{(1+z/n)^{n+z/2}} \quad (1.Prod.3)$$

Recurrence:

$$\frac{z!}{(z-1)!} = \frac{e^z}{\sqrt{2\pi}} e^{\int_{z-1}^z \ln t! dt} = \frac{e^z}{\sqrt{2\pi}} \lim_{n \rightarrow \infty} \left(\frac{(nz)!(2\pi)^{\frac{n-1}{2}}}{n^{nz+1/2}} \right)^{1/n} = z^z, \quad (1.Rec)$$

so for positive integer n ,

$$n! = 1^1 2^2 \dots n^n.$$

First pi:

$$\pi_1 = (-1/2)!^8 = 2^{2/3} \pi e^{\gamma-1 - \frac{\zeta'(2)}{\zeta(2)}} \quad (1.Pi)$$

$$= 5.77769 41333 41556 88333 68099 49818 72681 \dots$$

Replication formula:

$$\prod_{i=0}^{n-1} (z - i/n)_1 = \frac{(nz)_1^{1/n} (2^{1/3} \pi_1)^{\frac{n-1/n}{12}}}{n \frac{(nz+1)z}{2} + \frac{1}{12n}} \quad (1.Rep)$$

Reflection formula:

$$\frac{z_1!}{(-z)_1} = \left(\frac{z}{2}\right)^z e^{-\int_0^z \ln(\sin \pi t) dt} \quad (1.Ref)$$

Asymptoty:

$$\begin{aligned} z_1! &= (2^{1/3} \pi_1 z)^{1/12} z^{\frac{z(z+1)}{2}} e^{-\frac{1}{4}z^2 + \frac{1}{12}z - \frac{B_4}{2 \cdot 3 \cdot 4} z^{-2} - \frac{B_6}{4 \cdot 5 \cdot 6} z^{-4} - \frac{B_8}{6 \cdot 7 \cdot 8} z^{-6} - \dots} \\ &= (2^{1/3} \pi_1 e z)^{1/12} z^{\frac{z(z+1)}{2}} e^{-z^2/4} \left(1 + \frac{1}{720z^2} + O(z^{-4})\right). \end{aligned} \quad (1.Asym)$$

Taylor series:

$$z_1! = \frac{e^{\frac{1}{2}z - \frac{\gamma-1}{1 \cdot 2} z^2 + \frac{\zeta(2)}{2 \cdot 3} z^3 - \frac{\zeta(3)}{3 \cdot 4} z^4 + \dots}}{(2\pi)^{z/2}} \quad (1.Tay.e)$$

$$= 1 + \frac{1 - \ln 2\pi}{2} z + \frac{(\ln 2\pi)^2 - 2(\ln 2\pi) - 4\gamma + 5}{8} z^2 + \dots \quad (1.Tay)$$

Catalan's constant:

$$\lambda = \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \dots = 2\pi \log \frac{(-3/4)_1!}{(-1/4)_1!}$$

Product identities:

$$\prod_{n \geq 1} n^{\frac{6}{\pi^2 n^2}} = \prod_{p=\text{prime}} p^{\frac{1}{p^2-1}} = \frac{\pi_1 e^{1-\gamma}}{2^{2/3} \pi} = e^{-\frac{\zeta'(2)}{\zeta(2)}}$$

$$\prod_{n \geq 1} (2n+1)^{(2n+1)^{-2}} = \left(\frac{\pi_1 e^{1-\gamma}}{2\pi} \right)^{2/3}$$

Superfactorial:

$$z! \stackrel{\text{def}}{=} \frac{z!^{z+1}}{z!} = \left(\frac{2^7 \pi^9}{e^3 \pi_1^6} \right)^{\frac{1}{36}} \prod_{n \geq 1} \frac{\sqrt{2\pi} \left(\frac{n}{e} \right)^n (n+1)^{\frac{z(z+1)}{2} + \frac{1}{12}}}{(n+z)! n^{\frac{z(z-1)}{2} - \frac{5}{12}}}$$

so for integer n ,

$$n! = 1! 2! \dots n!$$

Asymptoty:

$$z! = \frac{z^{\frac{z^2}{2} + z + \frac{5}{12}} 2^{\frac{z}{2} + \frac{17}{36}} \pi^{\frac{z+1}{2}}}{\pi_1^{\frac{1}{12}} e^{\frac{3z^2}{4} + z}} \left(1 + \frac{1}{12z} + O(z^{-2}) \right). \quad (\text{S.Asym})$$

Determinants: Generalized Hilbert

$$\det \left(\frac{1}{i+j+b} \right)_n = \frac{(n+b)!^2 (n-1)!^2}{b! (2n+b)!} \quad (1.\text{det.1})$$

Vandermonde, discriminant (special case)

$$\det((ai+b)^{j+c})_n = a^{\frac{n(n+2c+1)}{2}} \left(\frac{(n+b/a)!}{b/a!} \right)^{c+1} (n-1)! \quad (1.\text{det.2})$$

(See *Knuth*, probs 1.2.3.37-38, and 5.1.4.9 for applications.)

Dilogarithm:

$$\operatorname{Li}_2(e^{2\pi iz}) = \pi^2 B_2(z) + 2\pi i \ln \frac{(z-1)_!}{(-z)_!}.$$

SECOND FACTORIAL

Definition:

$$z_2^! \stackrel{\text{def}}{=} \left(\frac{e^{3(z+\frac{1}{2})(z+1)}}{2\pi_1^3} \right)^{\frac{z}{18}} e^2 \int_0^z \ln t_1^! dt \quad (2.\text{Def}.0)$$

$$= \frac{e^{2 \int_0^z (z-t) \ln t^! dt - \frac{\zeta'(2)}{6}(\gamma - \frac{\zeta'(2)}{6}) + \frac{z(z+1)(2z+1)}{4}}{(2\pi)^{\frac{z(3z+1)}{6}}} \quad (2.\text{Def}.1)$$

Product definitions:

$$z_2^! = \left(\frac{e^{3((6-4\gamma)z^2+9z+1)}}{2^{2(9z+1)} \pi^{18z} \pi_1^6} \right)^{\frac{z}{36}} \prod_{n \geq 1} \frac{e^{nz+3z^2/2+z^3/3n}}{(1+\frac{z}{n})^{(n+z)^2}} \quad (2.\text{Prod}.0)$$

$$= (2\pi e^{2z-1})^{\frac{z(z-1)}{4}} \prod_{n \geq 1} \frac{(1+\frac{1}{n})^z e^{\frac{12(n+1)^2+18(z-1)(n+1)+(z-1)(4z-5)}{12}}}{(1+\frac{z}{n})^{(n+z)^2}} \quad (2.\text{Prod}.1)$$

Recurrence:

$$\frac{z_2^!}{(z-1)_2^!} = \left(\frac{e^{9z}}{2\pi_1^3} \right)^{\frac{z}{18}} e^2 \int_{z-1}^z \ln t_1^! dt = \left(\frac{e^{9z}}{2\pi_1^3} \right)^{\frac{z}{18}} \lim_{n \rightarrow \infty} \left(\frac{(nz)_1^{\frac{1}{n}} (2^{\frac{1}{2}} \pi_1)^{\frac{n-1/n}{12}}}{n^{\frac{(nz+1)z}{2} + \frac{1}{12n}}} \right)^{\frac{1}{n}} = z^{z^2}, \quad (2.\text{rec})$$

so for positive integer n ,

$$n_2^! = 1^{1^2} 2^{2^2} \dots n^{n^2}.$$

Second pi:

$$\pi_2 = \left(-\frac{1}{2}\right)_2^! = e^{\frac{7}{8}\zeta(3)\pi^{-2}} = 1.11245 53503 14827 97281 62913 28755 33992 32213 1868\dots$$

Asymptoty:

$$\begin{aligned} z_2^! &\simeq \pi_2^{\frac{z}{2}} z^{\frac{z(z+1)(2z+1)}{6}} e^{-\frac{1}{9}z^3 + \frac{1}{12}z + 2(\frac{B_4}{1 \cdot 2 \cdot 3 \cdot 4}z^{-1} + \frac{B_6}{3 \cdot 4 \cdot 5 \cdot 6}z^{-3} + \dots)} \\ &= \pi_2^{\frac{z}{2}} z^{\frac{z(z+1)(2z+1)}{6}} e^{\frac{\pi}{36}(3-4z^2)} \left(1 - \frac{1}{360z} + \frac{1}{259200z^2} + \frac{259193}{1959552000z^3} + \dots\right) \end{aligned} \quad (2.\text{Asym})$$

Replication formula:

$$\prod_{i=0}^{n-1} \left(z - \frac{i}{n}\right)_2^! = \left(\frac{(nz)_2^! \pi_2^{\frac{2(n^3-1)}{7}}}{n^{\frac{nz(nz+1)(2nz+1)}{6}}} \right)^{\frac{1}{n^2}}$$

Reflection formula:

$$z_2^! (-z)_2^! = \left(\frac{z}{2}\right)^{z^2} e^2 \int_0^z (t-z) \ln \sin \pi t dt$$

Taylor series:

$$z_2^! = \frac{e^{\frac{1}{12}z + \frac{3}{4}z^2 + (\frac{1}{2} - \frac{7}{3})z^3 + 2(\frac{\zeta(2)}{2 \cdot 3 \cdot 4}z^4 - \frac{\zeta(3)}{3 \cdot 4 \cdot 5}z^5 + \dots)}}{(2^{\frac{z}{2}} + \frac{1}{18} \pi^{\frac{z}{2}} \pi_1^{\frac{1}{6}})z} \quad (2.\text{Tay}.e)$$

$$= 1 + gz + \frac{2g^2 - 12l + 3}{4}z^2 + \frac{2g^3 - 9g(4l - 1) - 4\gamma + 6}{12}z^3 + \dots, \quad (2.\text{Tay})$$

$$\text{where } l = \frac{\log 2\pi}{6}, \quad g = \frac{1}{4} - \frac{1}{6}(\gamma + \log 2\pi - \frac{\zeta'(2)}{\zeta(2)}).$$

SECOND FACTORIAL

Definition:

$$z_2^! \stackrel{\text{def}}{=} \left(\frac{e^{3(z+\frac{1}{2})(z+1)}}{2\pi_1^3} \right)^{\frac{z}{18}} e^{2 \int_0^z \ln t_1^! dt} \quad (2.\text{Def.0})$$

$$= \frac{e^{2 \int_0^z (z-t) \ln t^! dt - \frac{z}{6}(\gamma - \frac{\zeta'(2)}{\zeta(2)}) + \frac{z(z+1)(2z+1)}{4}}}{(2\pi)^{\frac{z(3z+1)}{6}}} \quad (2.\text{Def.1})$$

Product definitions:

$$z_2^! = \left(\frac{e^{3((6-4\gamma)z^2+9z+1)}}{2^2(9z+1)\pi^{18z}\pi_1^6} \right)^{\frac{z}{36}} \prod_{n \geq 1} \frac{e^{nz+3z^2/2+z^3/3n}}{(1+\frac{z}{n})^{(n+z)^2}} \quad (2.\text{Prod.0})$$

$$= (2\pi e^{2z-1})^{\frac{z(z-1)}{4}} \prod_{n \geq 1} \frac{(1+\frac{1}{n})^{z((n+\frac{1}{2})(n+\frac{3}{2}z)+\frac{1}{3}(z-\frac{1}{2})(z+\frac{1}{2}))}}{(1+\frac{z}{n})^{(n+z)^2}} \quad (2.\text{Prod.1})$$

Recurrence:

$$\frac{z_2^!}{(z-1)_2^!} = \left(\frac{e^{9z}}{2\pi_1^3} \right)^{\frac{z}{18}} e^{2 \int_{z-1}^z \ln t_1^! dt} = \left(\frac{e^{9z}}{2\pi_1^3} \right)^{\frac{z}{18}} \lim_{n \rightarrow \infty} \left(\frac{(nz)_1^{\frac{1}{n}} (2^{\frac{1}{3}}\pi_1)^{\frac{n-1/n}{12}}}{n^{\frac{(nz+1)z}{2} + \frac{1}{12n}}} \right)^{\frac{1}{n}} = z^{z^2}, \quad (2.\text{rec})$$

so for positive integer n ,

$$n_2^! = 1^{1^2} 2^{2^2} \dots n^{n^2}.$$

Second pi:

$$\pi_2 = \left(-\frac{1}{2}\right)_2^! = e^{\frac{7}{3}\zeta(3)\pi^{-2}} = 1.11245\ 53503\ 14827\ 97281\ 62913\ 28755\ 33992\ 32213\ 1868\dots$$

Asymptoty:

$$z_2^! \simeq \pi_2^{\frac{z}{2}} z^{\frac{z(z+1)(2z+1)}{6}} e^{-\frac{1}{6}z^3 + \frac{1}{12}z + 2(\frac{B_4}{1.2.3.4}z^{-1} + \frac{B_6}{3.4.5.6}z^{-3} + \dots)} \\ = \pi_2^{\frac{z}{2}} z^{\frac{z(z+1)(2z+1)}{6}} e^{\frac{z}{36}(3-4z^2)} \left(1 - \frac{1}{360z} + \frac{1}{259200z^2} + \frac{259193}{1959552000z^3} + \dots\right) \quad (2.\text{Asym})$$

Replication formula:

$$\prod_{i=0}^{n-1} \left(z - \frac{i}{n}\right)_2^! = \left(\frac{(nz)_2^! \pi_2^{\frac{2(n^3-1)}{7}}}{n^{\frac{nz(nz+1)(2nz+1)}{6}}} \right)^{\frac{1}{n^2}}$$

Reflection formula:

$$z_2^!(-z)_2^! = \left(\frac{z}{2}\right)^{z^2} e^{2 \int_0^z (t-z) \ln \sin \pi t dt}$$

Taylor series:

$$z_2^! = \frac{e^{\frac{1}{12}z + \frac{3}{4}z^2 + (\frac{1}{2} - \frac{7}{3})z^3 + 2(\frac{\zeta(2)}{2.3.4}z^4 - \frac{\zeta(3)}{3.4.5}z^5 + \dots)}}{(2^{\frac{z}{2}} + \frac{1}{18}\pi^{\frac{z}{2}}\pi_1^{\frac{1}{6}})^z} \quad (2.\text{Tay.e})$$

$$= 1 + gz + \frac{2g^2 - 12l + 3}{4}z^2 + \frac{2g^3 - 9g(4l-1) - .4\gamma + 6}{12}z^3 + \dots, \quad (2.\text{Tay})$$

$$\text{where } l = \frac{\log 2\pi}{6}, \quad g = \frac{1}{4} - \frac{1}{6}(\gamma + \log 2\pi - \frac{\zeta'(2)}{\zeta(2)}).$$

NTH FACTORIAL

n recurrence:

$$z_n^! = \frac{e^{\frac{B_{n+1}(z+1) - B_{n+1}(1)}{n(n+1)} + n \int_0^z \ln t_n \cdot \frac{1}{t} dt}}{\left(\left(-\frac{1}{2}\right)_{n-1} n 2^{n-1} 2^{B_n(1)}\right)^{\frac{z}{2^n - 1}}}$$

z recurrence:

$$\frac{z_n^!}{(z-1)_n^!} = z^{z^n}$$

so for positive integer z ,

$$z_n^! = 1^{1^n} 2^{2^n} \dots z^{z^n}$$

Asymptoty:

$$\begin{aligned} \ln(z+y)_p^! &= \frac{\sum_{k \geq 0} \binom{p+1}{k} B_k(1+y) z^{p-k+1} (\ln z + \Psi(p+1) - \Psi(p-k+2)) + \frac{B_{p+1}(1) \ln 2}{2^{p+1} - 1}}{p+1} \\ &\quad + \frac{\ln(-\frac{1}{2})_p^!}{2 - 2^{-p}}, \quad p \neq \lfloor p \rfloor \\ &= \left(B_{p+1}(1+z+y) \ln z + \sum_{k=0}^{p+1} \binom{p+1}{k} B_k(1+y) z^{p-k+1} (\Psi(p+1) - \Psi(p-k+2)) \right) \\ &\quad - \sum_{k \geq p+1} \frac{B_{k+1}(1+y)(-z)^{p-k}}{(k+1) \binom{k}{p+1}} + \frac{B_{p+1}(1) \ln 2}{2^{p+1} - 1} (p+1)^{-1} + \frac{\ln(-\frac{1}{2})_p^!}{2 - 2^{-p}}, \quad p = \lfloor p \rfloor \end{aligned}$$

Useful fact:

$$\sum_{i=0}^{q-1} B_{k+1}\left(1+x - \frac{i}{q}\right) = q^{-k} B_{k+1}(1+qx)$$

Replication formula:

$$\prod_{i=0}^{q-1} \left(z - \frac{i}{q}\right)_p^! = \left(\frac{(qz)_p^! \left(2^{\frac{B_{p+1}(1)}{p+1}} \left(-\frac{1}{2}\right)_p^{2^p}\right)^{\frac{q^{p+1}-1}{2^{p+1}-1}}}{q^{\frac{B_{p+1}(1+qz)}{p+1}}} \right)^{q^{-p}},$$

$$(-)^{n+1} n \binom{x}{n} \prod_{k=0}^n \begin{pmatrix} \frac{k-n+1}{k+2} & (k+1)^p & 0 \\ 0 & \frac{k-n}{k+1} & \frac{x-n}{x-k} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{B_{p+1}(x+1) - B_p}{p+1} \\ 0 & 0 & -n \\ 0 & 0 & (-)^{n+1} n \binom{x}{n} \end{pmatrix},$$

$$n \prod_{k=1}^n \begin{pmatrix} \frac{k-n}{k+1} & \frac{k-n}{k+1} k^p & k^{p-1} \\ 0 & \frac{k-n}{k+1} & \frac{1}{k} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & B_p(1) \\ 0 & 0 & H_n \\ 0 & 0 & n \end{pmatrix},$$

i.e.,

$$\sum_{k=1}^n \frac{(-)^{k-1}}{k} \binom{n}{k} \sum_{j=1}^k j^p = B_p(1), \quad \forall n = [n] > p = [p]$$

$$\frac{\prod_{k \geq 1} 2t^{2^{-k}} - 1}{t \ln t}$$

$$\left(-\frac{1}{4}\right)! = 2^{1/4} \left(-\frac{1}{4}\right)!^{3/4} e^{-2/4\pi}$$

$$n! = 1 \cdot 2 \cdot 3 \cdots n.$$

$$n!_x := 1^{1^x} 2^{2^x} 3^{3^x} \cdots n^{n^x}.$$

$$z! \left(\frac{2\pi}{e^{z+1}}\right)^{z/2} = e^{\int_0^z \ln t! dt}$$

$$= z!^{z/2} \prod_{n \geq 1} \frac{e^z}{\left(1 + \frac{z}{n}\right)^{n+z/2}}.$$