[illegible]NYX	CHELTENHAM BOLD	GRAPHIQUE	BROADWAY	EMPIRE	THOROWGOOD	DELPHINIAN	ULTRA BODONI ITALIC	UMBRA
WINDSOR	PRISMA	PISTILLI ROMAN	THUNDERBIRD	COOPER BLACK ITALIC	GRECO ADORNADO	BETON EXTRA BOLD	NEULAND LICHTE	GOLD RUSH
FONTANESI	CORVINUS BOLD	CLEARFACE BOLD	CLARENDON BOLD	BETON OPEN	METROPOLIS BOLD	BULMER ITALIC	STYMIE EXTRA BOLD	WINDSOR OUTLINE
CRAW MODERN	BANK NOTE ITALIC	NORMANDE ITALIC	KISMET	THOROWGOOD ITALIC	ATHENAEUM	DAVIDA BOLD	ROYAL	STENCIL
OLYMPIC	CARTOON BOLD	CHELTENHAM OPEN	PEIGNOT BOLD	CICERO	CENTURY SCHOOLBOOK	COMMERCIAL SCRIPT	AMELIA	ALFERETA
DEUTCH BLACK	VISA	COMSTOCK	ANGLO	FUTURA BLACK	BOOKMAN ITALIC	MANDARIN	GALLIA	HESS NEOBOLD
COCHIN OLD STYLE	BANCO	CLOISTER BLACK	ARBORET	BODONI BOLD ITALIC	EUROSTILE CONDENSED	CHISEL	SMOKE	ST. CLAIR
SANS SERIF SHADED	AESTHETIC	BRITTANIC	MARBLEHEART	PROFIL	SAPHIR	CIRCUS	PLAYBILL	CELTIC ORNATE
KARNAC	GIORGIO	AGENCY	JANA	SHADOW	LATIN WIDE	CASLON OLD FACE	OLD BOWERY	TEA CHEST
SCOTFORD UNCIAL	FLIRT	EGYPTIAN OPEN	JIM CROW	SPHINX INLINE	P. T. BARNUM	EGYPTIAN CONDENSED	NORMANDIA OUTLINE	COLUMBUS
TRUMP GRAVURE	THORNE SHADED	BASKERVILLE BOLD	COOPER BLACK	BAKER SIGNET	LIBRA	PALATINO	FORUM II	FANTAIL

(ZEROTH) FACTORIAL

Product definitions:

$$z! = z!_0 = e^{-\gamma z}\prod_{n\geq 1}\frac{e^{z/n}}{1+z/n}\,, \qquad \text{(0.Prod.1)}$$

$$= \prod_{n\geq 1}\frac{(n+1)^z}{(n+z)n^{z-1}} = \prod_{n\geq 1}\frac{(1+1/n)^z}{1+z/n}\,. \qquad \text{(0.Prod.2)}$$

Recurrence:

$$\frac{z!}{(z-1)!} = z\,. \qquad \text{(0.Rec)}$$

Pi:

$$\pi = \pi_0 = (-1/2)!^2 = 3.14159\ 26535\ 89793\ \ldots\ . \qquad \text{(0.Pi)}$$

Replication formula:

$$\prod_{i=0}^{n-1}(z - i/n)! = \frac{(nz)!(2\pi)^{\frac{n-1}{2}}}{n^{nz+1/2}}\,. \qquad \text{(0.Rep)}$$

Reflection formula:

$$z!(-z)! = \frac{\pi z}{\sin \pi z}\,. \qquad \text{(0.Ref)}$$

Asymptosy:

$$z! = \sqrt{2\pi z}\, z^z\, e^{-z + \frac{B_2}{1\cdot 2}z^{-1} + \frac{B_4}{3\cdot 4}z^{-3} + \frac{B_6}{5\cdot 6}z^{-5} + \cdots}$$

$$= \sqrt{2\pi z}\, z^z\, e^{-z + \frac{1}{12}z^{-1} - \frac{1}{360}z^{-3} + \frac{1}{1260}z^{-5} - \cdots}$$

$$= \sqrt{2\pi z}\left(\frac{z}{e}\right)^z\left(1 + \frac{1}{12z} + \frac{1}{288z^2} - \frac{139}{51840z^3} - O(z^{-4})\right) \qquad \text{(0.Asym)}$$

$$= \sqrt{2\pi z}\left(\frac{z}{e}\right)^z\prod_{n\geq z}\frac{(1+1/n)^{n+1/2}}{e}\,.$$

Product of the general rational function: if $-a_1, -a_2, \ldots, -a_k$ are the poles and roots and $p_1, p_2, \ldots, p_k$ are the respective multiplicities (counting poles negatively), then, since (0.Prod.2) is equivalent to

$$\prod_{n\geq 1} \frac{(n+a_1)^{p_1}(n+a_2)^{p_2}\cdots(n+a_k)^{p_k}}{n^{p_1+p_2+\cdots+p_k}}\left(\frac{n}{n+1}\right)^{a_1p_1+a_2p_2+\cdots+a_kp_k} = \frac{1}{a_1!^{p_1}a_2!^{p_2}\cdots a_k!^{p_k}}, \tag{0.Genp}$$

the desired product converges, and to this limit, iff $\Sigma p_i = \Sigma a_i p_i = 0$.

Faster convergence: (0.Prod.1) is the special case $k = 1$, $m \to \infty$, of

$$\frac{\displaystyle\prod_{n=1}^{m} \frac{e^{\frac{z}{n} - \frac{1}{2}(\frac{z}{n})^2 + \frac{1}{3}(\frac{z}{n})^3 - \cdots - \frac{1}{k}(-\frac{z}{n})^k}}{1+z/n}}{e^{\gamma z - \frac{\zeta(2)}{2}z^2 + \frac{\zeta(3)}{3}z^3 - \cdots - \frac{\zeta(k)}{k}(-z)^k}} = z!\left(1 - \frac{(-z)^{k+1}}{k(k+1)m^k} + O(\frac{z^{k+2}}{k^2 m^{k+1}})\right), \tag{0.Fast.1}$$

where the exponent in the multiplicand is just k terms of $\log 1+z/n$.

Similarly, (0.Prod.2) is the special case $k = 0$, $m \to \infty$, of

$$\frac{\displaystyle\prod_{n=1}^{m} \frac{(n+1)^{\frac{z+k}{k+1}}}{n^{\frac{z-1}{k+1}}} \prod_{i=0}^{k} (n+z+i)^{\frac{(-1)^{i+1}(z-1)z\ldots(z+i-2)(z+i+1)\ldots(z+k)}{i!(k-i)!(k+1)}}}{\displaystyle\prod_{i=1}^{k}((z+1)(z+2)\ldots(z+i))^{\frac{(-1)^{i}(z-1)z\ldots(z+i-2)(z+i+1)\ldots(z+k)}{i!(k-i)!(k+1)}}} \tag{0.Fast.2}$$

$$= z!\left(1 - \frac{(z-1)\,z\ldots(z+k)}{(k+1)^2(k+2)m^{k+1}} + O(\frac{1}{k^3 m^{k+2}})\right).$$

Taylor series: sending $k \to \infty$ in (0.Fast.1),

$$z! = e^{-\gamma z + \frac{\zeta(2)}{2}z^2 - \frac{\zeta(3)}{3}z^3 + \frac{\zeta(4)}{4}z^4 - \cdots} \tag{0.Tay.e}$$

$$= 1 - \gamma z + \frac{6\gamma^2+\pi^2}{12}z^2 - \frac{2\gamma^3+\gamma\pi^2+4\zeta(3)}{12}z^3 + \cdots \tag{0.Tay}$$

$$\frac{(z-1/2)!}{\sqrt{\pi}} = 4^{-z} e^{-\gamma z + \frac{3\zeta(2)}{2}z^2 - \frac{7\zeta(3)}{3}z^3 + \frac{15\zeta(4)}{4}z^4 - \cdots} \tag{0.Tay.e.5}$$

$$= 1 - gz + \frac{2g^2+\pi^2}{4}z^2 - \frac{2g^3+3\pi^2 g+28\zeta(3)}{12}z^3 + \cdots, \tag{0.Tay.5}$$

where $g = \gamma + 2\ln 2$.

FIRST FACTORIAL

Definition:

$$z!_1 = \frac{e^{\int_0^z \ln t!\,dt + \frac{z(z+1)}{2}}}{(2\pi)^{z/2}} . \qquad \text{(1.Def)}$$

Product definitions:

$$z!_1 = \left(\frac{e^{\frac{z+\gamma z+1}{2}} z!}{\sqrt{2\pi}}\right)^z \prod_{n\geq 1} \frac{e^{z-z^2/2n}}{(1+z/n)^n} \qquad \text{(1.Prod.0)}$$

$$= \frac{e^{\frac{(z-\gamma z+1)z}{2}}}{(2\pi)^{z/2}} \prod_{n\geq 1} \frac{e^{z+z^2/2n}}{(1+z/n)^{n+z}} \qquad \text{(1.Prod.1)}$$

$$= e^{\frac{(z-1)z}{2}} \prod_{n\geq 1} \frac{(1+1/n)^{(n+\frac{z+1}{2})z}}{(1+z/n)^{n+z}} \qquad \text{(1.Prod.2)}$$

$$= \left(\frac{e^{z+1} z!}{2\pi}\right)^{z/2} \prod_{n\geq 1} \frac{e^z}{(1+z/n)^{n+z/2}} . \qquad \text{(1.Prod.3)}$$

Recurrence:

$$\frac{z!_1}{(z-1)!_1} = \frac{e^z}{\sqrt{2\pi}} e^{\int_{z-1}^z \ln t!\,dt} = \frac{e^z}{\sqrt{2\pi}} \lim_{n\to\infty}\left(\frac{(nz)!(2\pi)^{\frac{n-1}{2}}}{n^{nz+1/2}}\right)^{1/n} = z^z , \qquad \text{(1.Rec)}$$

so for positive integer n,

$$n!_1 = 1^1 2^2 \ldots n^n .$$

First pi:

$$\pi_1 = (-1/2)!_1^{\,8} = 2^{2/3}\,\pi\, e^{\gamma - 1 - \frac{\zeta'(2)}{\zeta(2)}} \qquad \text{(1.Pi)}$$

$$= 5.77769\ 41333\ 41556\ 88333\ 68099\ 49818\ 72681\ \ldots .$$

Replication formula:

$$\prod_{i=0}^{n-1}(z - i/n)!_1 = \frac{(nz)!_1^{1/n}\,(2^{1/3}\pi_1)^{\frac{n-1/n}{12}}}{n^{\frac{(nz+1)z}{2}+\frac{1}{12n}}}\,. \qquad \text{(1.Rep)}$$

Reflection formula:

$$\frac{z!_1}{(-z)!_1} = \left(\frac{z}{2}\right)^{z} e^{-\int_0^z \ln(\sin \pi t)\,dt}\,. \qquad \text{(1.Ref)}$$

Asymptosy:

$$\begin{aligned} z!_1 &= (2^{1/3}\pi_1 z)^{1/12}\, z^{\frac{z(z+1)}{2}}\, e^{-\frac{1}{4}z^2 + \frac{1}{12} - \frac{B_4}{2\cdot3\cdot4}z^{-2} - \frac{B_6}{4\cdot5\cdot6}z^{-4} - \frac{B_8}{6\cdot7\cdot8}z^{-6} - \cdots} \\ &= (2^{1/3}\pi_1 e z)^{1/12}\, z^{\frac{z(z+1)}{2}}\, e^{-z^2/4}\left(1 + \frac{1}{720z^2} + O(z^{-4})\right). \end{aligned} \qquad \text{(1.Asym)}$$

Taylor series:

$$z!_1 = \frac{e^{\frac{1}{2}z - \frac{\gamma-1}{1\cdot2}z^2 + \frac{\zeta(2)}{2\cdot3}z^3 - \frac{\zeta(3)}{3\cdot4}z^4 + \cdots}}{(2\pi)^{z/2}} \qquad \text{(1.Tay.e)}$$

$$= 1 + \frac{1-\ln 2\pi}{2}z + \frac{(\ln 2\pi)^2 - 2(\ln 2\pi) - 4\gamma + 5}{8}z^2 + \cdots \qquad \text{(1.Tay)}$$

Catalan's constant:

$$\lambda = \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \cdots = 2\pi \log \frac{(-3/4)!_1}{(-1/4)!_1}$$

Product identities:

$$\prod_{n\ge1} n^{\frac{6}{\pi^2 n^2}} = \prod_{p=\text{prime}} p^{\frac{1}{p^2-1}} = \frac{\pi_1 e^{1-\gamma}}{2^{2/3}\pi} = e^{-\frac{\zeta'(2)}{\zeta(2)}}$$

$$\prod_{n\ge1}(2n+1)^{(2n+1)^{-2}} = \left(\frac{\pi_1 e^{1-\gamma}}{2\pi}\right)^{\pi^2/8}$$

Superfactorial:

$$z^{!} \overset{\text{def}}{=} \frac{z!^{z+1}}{z^{!}_{1}} = \left(\frac{2^7\pi^9}{e^3\pi_1^6}\right)^{\frac{1}{36}} \prod_{n\ge1} \frac{\sqrt{2\pi}\left(\frac{n}{e}\right)^n (n+1)^{\frac{z(z+1)}{2}+\frac{1}{12}}}{(n+z)!\, n^{\frac{z(z-1)}{2}-\frac{5}{12}}},$$

so for integer n,

$$n^{!} = 1!\,2!\ldots n!\,.$$

Asymptosy:

$$z^{!} = \frac{z^{\frac{z^2}{2}+z+\frac{5}{12}}\, 2^{\frac{z}{2}+\frac{17}{36}}\, \pi^{\frac{z+1}{2}}}{\pi_1^{\frac{1}{12}}\, e^{\frac{3z^2}{4}+z}}\left(1+\frac{1}{12z}+O(z^{-2})\right). \qquad \text{(S.Asym)}$$

Determinants: Generalized Hilbert

$$\det\left(\frac{1}{i+j+b}\right)_n = \frac{(n+b)^{!2}(n-1)^{!2}}{b^{!}\,(2n+b)^{!}} \qquad \text{(1.det.1)}$$

Vandermonde, discriminant (special case)

$$\det((ai+b)^{j+c})_n = a^{\frac{n(n+2c+1)}{2}}\left(\frac{(n+b/a)!}{b/a!}\right)^{c+1}(n-1)^{!} \qquad \text{(1.det.2)}$$

(See *Knuth*, probs 1.2.3.37-38, and 5.1.4.9 for applications.)

Dilogarithm:

$$\mathrm{Li}_2(e^{2\pi i z}) = \pi^2 B_2(z) + 2\pi i \ln \frac{(z-1)!_1}{(-z)!_1}.$$

SECOND FACTORIAL

Definition:

$$z!_2 \overset{\text{def}}{=} \left(\frac{e^{3(z+\frac{1}{2})(z+1)}}{2\pi_1^3}\right)^{\frac{z}{18}} e^{2\int_0^z \ln t!_1\, dt} \tag{2.Def.0}$$

$$= \frac{e^{2\int_0^z (z-t)\ln t!\,dt - \frac{z}{6}(\gamma - \frac{\zeta'(2)}{\zeta(2)}) + \frac{z(z+1)(2z+1)}{4}}}{(2\pi)^{\frac{z(3z+1)}{6}}} \tag{2.Def.1}$$

Product definitions:

$$z!_2 = \left(\frac{e^{3((6-4\gamma)z^2+9z+1)}}{2^{2(9z+1)}\pi^{18z}\pi_1^6}\right)^{\frac{z}{36}} \prod_{n\geq 1} \frac{e^{nz+3z^2/2+z^3/3n}}{(1+\frac{z}{n})^{(n+z)^2}} \tag{2.Prod.0}$$

$$= (2\pi e^{2z-1})^{\frac{z(z-1)}{4}} \prod_{n\geq 1} \frac{(1+\frac{1}{n})^{z\frac{12(n+1)^2+18(z-1)(n+1)+(z-1)(4z-5)}{12}}}{(1+\frac{z}{n})^{(n+z)^2}} \tag{2.Prod.1}$$

Recurrence:

$$\frac{z!_2}{(z-1)!_2} = \left(\frac{e^{9z}}{2\pi_1^3}\right)^{\frac{z}{18}} e^{2\int_{z-1}^z \ln t!_1\, dt} = \left(\frac{e^{9z}}{2\pi_1^3}\right)^{\frac{z}{18}} \lim_{n\to\infty}\left(\frac{(nz)!_1^{\frac{1}{n}}(2^{\frac{1}{3}}\pi_1)^{\frac{n-1/n}{12}}}{n^{\frac{(nz+1)z}{2}+\frac{1}{12n}}}\right)^{\frac{1}{n}} = z^{z^2}, \tag{2.rec}$$

so for positive integer n,

$$n!_2 = 1^{1^2}2^{2^2}\ldots n^{n^2}.$$

Second pi:

$$\pi_2 = (-\frac{1}{2})!_2^{\,2} = e^{\frac{7}{8}\zeta(3)\pi^{-2}} = 1.11245\,53503\,14827\,97281\,62913\,28755\,33992\,32213\,1868\ldots$$

Asymptosy:

$$z!_2 \simeq \pi_2^{\frac{2}{7}} z^{\frac{z(z+1)(2z+1)}{6}} e^{-\frac{1}{9}z^3+\frac{1}{12}z+2(\frac{B_4}{1\cdot 2\cdot 3\cdot 4}z^{-1}+\frac{B_6}{3\cdot 4\cdot 5\cdot 6}z^{-3}+\cdots)}$$

$$= \pi_2^{\frac{2}{7}} z^{\frac{z(z+1)(2z+1)}{6}} e^{\frac{z}{36}(3-4z^2)}\left(1 - \frac{1}{360z} + \frac{1}{259200z^2} + \frac{259193}{1959552000z^3} + \cdots\right) \tag{2.Asym}$$

Replication formula:

$$\prod_{i=0}^{n-1}(z-\frac{i}{n})!_2 = \left(\frac{(nz)!_2\pi_2^{\frac{2(n^3-1)}{7}}}{n^{\frac{nz(nz+1)(2nz+1)}{6}}}\right)^{\frac{1}{n^2}}$$

Reflection formula:

$$z!_2(-z)!_2 = (\frac{z}{2})^{z^2} e^{2\int_0^z (t-z)\ln\sin\pi t\, dt}$$

Taylor series:

$$z!_2 = \frac{e^{\frac{1}{12}z+\frac{3}{4}z^2+(\frac{1}{2}-\frac{\gamma}{3})z^3+2(\frac{\zeta(2)}{2\cdot 3\cdot 4}z^4-\frac{\zeta(3)}{3\cdot 4\cdot 5}z^5+\cdots)}}{(2^{\frac{z}{2}+\frac{1}{18}}\pi^{\frac{z}{2}}\pi_1^{\frac{1}{6}})^z} \tag{2.Tay.e}$$

$$= 1 + gz + \frac{2g^2-12l+3}{4}z^2 + \frac{2g^3-9g(4l-1)-4\gamma+6}{12}z^3 + \cdots, \tag{2.Tay}$$

where $l = \frac{\log 2\pi}{6}$, $g = \frac{1}{4} - \frac{1}{6}(\gamma + \log 2\pi - \frac{\zeta'(2)}{\zeta(2)})$.

SECOND FACTORIAL

Definition:

$$z_2^! \overset{\text{def}}{=} \left(\frac{e^{3(z+\frac{1}{2})(z+1)}}{2\pi_1^3}\right)^{\frac{z}{18}} e^{2\int_0^z \ln t_1^! \, dt} \tag{2.Def.0}$$

$$= \frac{e^{2\int_0^z (z-t)\ln t!\,dt - \frac{z}{6}(\gamma - \frac{\zeta'(2)}{\zeta(2)}) + \frac{z(z+1)(2z+1)}{4}}}{(2\pi)^{\frac{z(3z+1)}{6}}} \tag{2.Def.1}$$

Product definitions:

$$z_2^! = \left(\frac{e^{3((6-4\gamma)z^2+9z+1)}}{2^{2(9z+1)}\pi^{18z}\pi_1^6}\right)^{\frac{z}{36}} \prod_{n\geq 1} \frac{e^{nz+3z^2/2+z^3/3n}}{(1+\frac{z}{n})^{(n+z)^2}} \tag{2.Prod.0}$$

$$= (2\pi e^{2z-1})^{\frac{z(z-1)}{4}} \prod_{n\geq 1} \frac{(1+\frac{1}{n})^{z\left((n+\frac{1}{2})(n+\frac{3}{2}z)+\frac{1}{3}(z-\frac{1}{2})(z+\frac{1}{2})\right)}}{\left(1+\frac{z}{n}\right)^{(n+z)^2}} \tag{2.Prod.1}$$

Recurrence:

$$\frac{z_2^!}{(z-1)_2^!} = \left(\frac{e^{9z}}{2\pi_1^3}\right)^{\frac{z}{18}} e^{2\int_{z-1}^z \ln t_1^! \, dt} = \left(\frac{e^{9z}}{2\pi_1^3}\right)^{\frac{z}{18}} \lim_{n\to\infty}\left(\frac{(nz)_1^{!\frac{1}{n}}(2^{\frac{1}{3}}\pi_1)^{\frac{n-1/n}{12}}}{n^{\frac{(nz+1)z}{2}+\frac{1}{12n}}}\right)^{\frac{1}{n}} = z^{z^2}, \tag{2.rec}$$

so for positive integer n,

$$n_2^! = 1^{1^2}2^{2^2}\ldots n^{n^2}.$$

Second pi:

$$\pi_2 = (-\frac{1}{2})_2^{!\,2} = e^{\frac{7}{8}\zeta(3)\pi^{-2}} = 1.11245\ 53503\ 14827\ 97281\ 62913\ 28755\ 33992\ 32213\ 1868\ldots$$

Asymptosy:

$$z_2^! \simeq \pi_2^{\frac{2}{7}} z^{\frac{z(z+1)(2z+1)}{6}} e^{-\frac{1}{9}z^3+\frac{1}{12}z+2(\frac{B_4}{1\cdot 2\cdot 3\cdot 4}z^{-1}+\frac{B_6}{3\cdot 4\cdot 5\cdot 6}z^{-3}+\cdots)}$$

$$= \pi_2^{\frac{2}{7}} z^{\frac{z(z+1)(2z+1)}{6}} e^{\frac{z}{36}(3-4z^2)}\left(1 - \frac{1}{360z} + \frac{1}{259200z^2} + \frac{259193}{1959552000z^3} + \cdots\right) \tag{2.Asym}$$

Replication formula:

$$\prod_{i=0}^{n-1}(z-\frac{i}{n})_2^! = \left(\frac{(nz)_2^!\pi_2^{\frac{2(n^3-1)}{7}}}{n^{\frac{nz(nz+1)(2nz+1)}{6}}}\right)^{\frac{1}{n^2}}$$

Reflection formula:

$$z_2^!(-z)_2^! = \left(\frac{z}{2}\right)^{z^2} e^{2\int_0^z (t-z)\ln\sin\pi t\,dt}$$

Taylor series:

$$z_2^! = \frac{e^{\frac{1}{12}z+\frac{3}{4}z^2+(\frac{1}{2}-\frac{7}{3})z^3+2(\frac{\zeta(2)}{2\cdot 3\cdot 4}z^4-\frac{\zeta(3)}{3\cdot 4\cdot 5}z^5+\cdots)}}{(2^{\frac{z}{2}+\frac{1}{18}}\pi^{\frac{z}{2}}\pi_1^{\frac{1}{6}})^z} \tag{2.Tay.e}$$

$$= 1 + gz + \frac{2g^2-12l+3}{4}z^2 + \frac{2g^3-9g(4l-1)-.4\gamma+6}{12}z^3 + \cdots, \tag{2.Tay}$$

where $l = \frac{\log 2\pi}{6}$, $g = \frac{1}{4} - \frac{1}{6}(\gamma + \log 2\pi - \frac{\zeta'(2)}{\zeta(2)})$.

NTH FACTORIAL

n recurrence:

$$z!_{n} = \frac{e^{\frac{B_{n+1}(z+1)-B_{n+1}(1)}{n(n+1)} + n\int_0^z \ln t_{n-1}^{!}\,dt}}{\left((-\frac{1}{2})_{n-1}^{!}{}^{n2^{n-1}} 2^{B_n(1)}\right)^{\frac{z}{2^n-1}}}$$

z recurrence:

$$\frac{z!_{n}}{(z-1)!_{n}} = z^{z^n}$$

so for positive integer z,

$$z!_{n} = 1^{1^n} 2^{2^n} \ldots z^{z^n}$$

Asymptosy:

$$\ln(z+y)!_{p} = \frac{\sum_{k\geq 0}\binom{p+1}{k}B_k(1+y)z^{p-k+1}\left(\ln z + \Psi(p+1) - \Psi(p-k+2)\right) + \frac{B_{p+1}(1)\ln 2}{2^{p+1}-1}}{p+1}$$
$$+ \frac{\ln(-\frac{1}{2})!_{p}}{2-2^{-p}}, \qquad p \neq \lfloor p \rfloor$$
$$= \left(B_{p+1}(1+z+y)\ln z + \sum_{k=0}^{p+1}\binom{p+1}{k}B_k(1+y)z^{p-k+1}\left(\Psi(p+1)-\Psi(p-k+2)\right)\right.$$
$$\left. - \sum_{k\geq p+1}\frac{B_{k+1}(1+y)(-z)^{p-k}}{(k+1)\binom{k}{p+1}} + \frac{B_{p+1}(1)\ln 2}{2^{p+1}-1}\right)(p+1)^{-1} + \frac{\ln(-\frac{1}{2})!_{p}}{2-2^{-p}}. \qquad p = \lfloor p \rfloor$$

Useful fact:

$$\sum_{i=0}^{q-1} B_{k+1}(1+x-\frac{i}{q}) = q^{-k}B_{k+1}(1+qx)$$

Replication formula:

$$\prod_{i=0}^{q-1}(z-\frac{i}{q})!_{p} = \left(\frac{(qz)!_{p}\left(2^{\frac{B_{p+1}(1)}{p+1}}(-\frac{1}{2})!_{p}{}^{2^p}\right)^{\frac{q^{p+1}-1}{2^{p+1}-1}}}{q^{\frac{B_{p+1}(1+qz)}{p+1}}}\right)^{q^{-p}},$$

$$(-)^{n+1} n \binom{x}{n} \prod_{k=0}^{n} \begin{pmatrix} \frac{k-n+1}{k+2} & (k+1)^p & 0 \\ 0 & \frac{k-n}{k+1} & \frac{x-n}{x-k} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{B_{p+1}(x+1)-B_p}{p+1} \\ 0 & 0 & -n \\ 0 & 0 & (-)^{n+1} n \binom{x}{n} \end{pmatrix},$$

$$n \prod_{k=1}^{n} \begin{pmatrix} \frac{k-n}{k+1} & \frac{k-n}{k+1} k^p & k^{p-1} \\ 0 & \frac{k-n}{k+1} & \frac{1}{k} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & B_p(1) \\ 0 & 0 & H_n \\ 0 & 0 & n \end{pmatrix},$$

i.e.,

$$\sum_{k=1}^{n} \frac{(-)^{k-1}}{k} \binom{n}{k} \sum_{j=1}^{k} j^p = B_p(1), \qquad \forall n = \lfloor n \rfloor > p = \lfloor p \rfloor$$

$$\frac{\prod_{k \geq 1} 2t^{2^{-k}} - 1}{t \ln t}$$

$$\left(-\tfrac{1}{4}\right) \underset{1}{!} = 2^{1/32} \left(-\tfrac{1}{4}\right) \underset{1}{!}^{3/4} e^{-G/4\pi}$$

$$n! = 1 \cdot 2 \cdot 3 \cdots n.$$

$$n \underset{x}{!} := 1^{1^x} 2^{2^x} 3^{3^x} \ldots n^{n^x}.$$

$$z \underset{1}{!} \left(\frac{2\pi}{e^{z+1}}\right)^{z/2} = e^{\int_0^z \ln t! \, dt}$$

$$= z!^{z/2} \prod_{n \geq 1} \frac{e^z}{\left(1 + \frac{z}{n}\right)^{n+z/2}}.$$