Let χ be a real Dirichlet character modulo N, we are interested in the value of

$$S(\chi) \coloneqq -\frac{1}{N}\sum_{i=1}^{N} i\chi(i).$$

If χ is an even character (i.e., $\chi(-1) = 1$) and $N > 1$, then

$$\sum_{i=1}^{N} i\chi(i) = \frac{1}{2}\left(\sum_{i=1}^{N} i\chi(i) + \sum_{i=0}^{N-1} i\chi(i)\right) = \frac{1}{2}\sum_{i=1}^{N}\big(i\chi(i) + (N-i)\chi(N-i)\big)$$

$$= \frac{1}{2}\sum_{i=1}^{N} N\chi(i),$$

$$S(\chi) = -\frac{1}{2}\sum_{i=1}^{N}\chi(i) = \begin{cases} -\dfrac{\varphi(N)}{2}, \text{if } N \text{ is a principal character} \\ \quad 0 \quad, \text{otherwise} \end{cases}.$$

So we'll suppose that χ is an odd character (i.e., $\chi(-1) = -1$).

Lemma 1. If χ_n is any Dirichlet character whose period is n, then for any k,

$$S(\chi) = -\frac{1}{kn}\sum_{i=1}^{kn} i\chi_n(i).$$

Proof.

$$\sum_{i=1}^{kn} i\chi_n(i) = \sum_{j=0}^{k-1}\sum_{i=1}^{n}(i + jn)\chi_n(i) = k\sum_{i=1}^{n} i\chi_n(i) + n\left(\sum_{j=0}^{k-1} j\right)\left(\sum_{i=1}^{n}\chi_n(i)\right)$$

$$= k\sum_{i=1}^{n} i\chi_n(i).$$

This result means that $S(\chi)$ is not affected by the upper bound of the summation index as long as it is divisible by the period of χ.

Lemma 2. Suppose χ_n to be any Dirichlet character whose period is n, χ_p be the principal Dirichlet character modulo p, where p is any prime coprime to n, then

$$S(\chi_n\chi_p) = \big(1 - \chi_n(p)\big)S(\chi_n).$$

Proof.

$$\sum_{i=1}^{np} i\chi_n(i)\chi_p(i) = \sum_{i=1}^{np} i\chi_n(i) - \sum_{j=1}^{n} jp\chi_n(jp)$$

$$= p\sum_{i=1}^{n} i\chi_n(i) - p\chi_n(p)\sum_{j=1}^{n} j\chi_n(j) \text{ (By Lemma 1)}$$

$$= p\big(1 - \chi_n(p)\big)\sum_{i=1}^{n} i\chi_n(i).$$

Lemma 3. Let χ be a character, then there exists a unique way to decompose χ into $\chi=\chi_0\chi_d$ and a pair of numbers (k,d), where χ_0 is a principal character modulo k, χ_d is a primitive character modulo d, k is squarefree, and $\gcd(k,d)=1$.

Proof. Let χ be a character modulo N, then χ can be uniquely decomposed into $\chi=\chi_{N1}\chi_{N2}\cdots\chi_{Nl}$, where χ_{Ni} is a character modulo $p_i^{\alpha_i}$, $p_1^{\alpha_1}p_2^{\alpha_2}\dots p_l^{\alpha_l}=N$. Let

$$\chi_0=\prod_{\chi_{Ni}\text{ is principal}}\chi_{Ni},\ \chi_d=\prod_{\chi_{Ni}\text{ is non-principal}}\chi_{Ni}.$$

Then $k=\prod_{\chi_{Ni}\text{ is principal}}p_i$. When χ_{Ni} is a non-principal character modulo $p_i^{\alpha_i}$, it is primitive modulo its own period, so χ_d is a primitive character modulo its period d. Obviously k is squarefree, and $\gcd(k,d)=1$.

If we have another pair (χ_0',χ_d') such that $\chi=\chi_0'\chi_d'$, χ_0' is a principal character modulo k', χ_d' is a primitive character modulo d', k' is squarefree, and $\gcd(k',d')=1$, write

$$\chi_0=\chi_{01}\chi_{02}\cdots\chi_{0r},\chi_d=\chi_{d1}\chi_{d2}\cdots\chi_{dt};$$
$$\chi_0'=\chi_{01}'\chi_{02}'\cdots\chi_{0r'}',\chi_d'=\chi_{d1}'\chi_{d2}'\cdots\chi_{dt'}',$$

where $\{\chi_{0i}\},\{\chi_{0i}'\}$ are principal characters modulo prime powers, $\{\chi_{dj}\},\{\chi_{dj}'\}$ are non-principal characters modulo prime powers, by the unique decomposition of χ into characters modulo prime powers, we have

$$\{\chi_{01},\chi_{02},\dots,\chi_{0r},\chi_{d1},\chi_{d2},\dots,\chi_{dt}\}=\{\chi_{01}',\chi_{02}',\dots,\chi_{0r'}',\chi_{d1}',\chi_{d2}',\dots,\chi_{dt'}'\},$$

$$\{\chi_{01},\chi_{02},\dots,\chi_{0r}\}=\{\chi_{01}',\chi_{02}',\dots,\chi_{0r'}'\},\{\chi_{d1},\chi_{d2},\dots,\chi_{dt}\}=\{\chi_{d1}',\chi_{d2}',\dots,\chi_{dt'}'\}.$$

This gives $\chi_0'=\chi_0$ and $\chi_d'=\chi_d$, so $k'=k$ and $d'=d$.

Theorem. Let χ be an odd real Dirichlet character modulo N. Decompose χ into $\chi=\chi_0\chi_d$, where χ_0 is a principal character modulo k, χ_d is a primitive character modulo d, k is squarefree, and $\gcd(k,d)=1$, then $-d$ is the fundamental discriminant of an imaginary quadratic number field, and

$$S(\chi)=\frac{2h(-d)}{w(-d)}\prod_{p\mid N,p\text{ prime}}\left(1-\left(\frac{-d}{p}\right)\right),$$

where $h(-d)$ is the class number of $K=\mathbb{Q}[\sqrt{-d}]$, and $w(-d)$ is the number of elements in K whose norms are 1, given by

$$w(-d)=\begin{cases}6, d=3\\4, d=4,\\2, d>4\end{cases}$$

and $\left(\frac{-d}{i}\right)$ is the Kronecker symbol.

Example. Let $N=21$,

$$\chi = \{1,1,0,1,-1,0,0,1,0,-1,1,0,-1,0,0,1,-1,0,-1,-1,0\}.$$

Then $\chi_0 = \{1,1,0\}$ is principal, $\chi_7 = \{1,1,-1,1,-1,-1,0\}$ is a primitive character modulo 7, and

$$S(\chi) = 2 = \frac{2h(-7)}{w(-7)}\left(1 - \left(\frac{-7}{3}\right)\right)\left(1 - \left(\frac{-7}{7}\right)\right).$$

Proof. Write

$$\chi_0 = \chi_{01}\chi_{02}\cdots\chi_{0r}, S(\chi) = S(\chi_0\chi_d) = S(\chi_d\chi_{01}\chi_{02}\cdots\chi_{0r}),$$

where $\{\chi_{0i}\}$ is the principal character modulo p_i, $p_1p_2\dots p_r = k$.

By Lemma 2,

$$S(\chi) = S(\chi_d)\prod_{i=1}^{r}\left(1 - \chi_d(p_i)\right).$$

Note that if $q|N$ but $q \neq p_i$, then $q|d$, $\chi_d(q) = 0$. So this formula can be simplified to

$$S(\chi) = S(\chi_d)\prod_{p|N, p \text{ prime}}\left(1 - \chi_d(p)\right).$$

There is a primitive character modulo d means either one of the following two statements holds:

(a) d is the fundamental discriminant of a real quadratic number field, and

$$\chi_d(i) = \left(\frac{d}{i}\right);$$

(b) $-d$ is the fundamental discriminant of an imaginary quadratic number field, and

$$\chi_d(i) = \left(\frac{-d}{i}\right).$$

Since χ is an odd character, we have $\chi_d(-1) = -1$, so we know statement (b) is true. Then we have the well-known class number formula:

$$S(\chi_d) = \frac{2h(-d)}{w(-d)},$$

So now we have

$$S(\chi) = \frac{2h(-d)}{w(-d)}\prod_{p|N, p \text{ prime}}\left(1 - \left(\frac{-d}{p}\right)\right),$$

which is the desired result.

Specially, let $\chi = \left(\frac{-D}{i}\right)$ be a real Dirichlet character modulo D for some $D \equiv 0,3 \pmod 4$, then d is the unique number such that $-d$ is the unique fundamental discriminant and D/d is a square. This gives

$$-\frac{1}{D}\sum_{i=1}^{D} i\left(\frac{-D}{i}\right) = S(\chi) = \frac{2h(-d)}{w(-d)}\prod_{p|D, p \text{ prime}}\left(1 - \left(\frac{-d}{p}\right)\right).$$