Proof of a conjecture stated in A325046

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Let

$$A(x)=\sum_{n=0}^{\infty}\frac{x^n(1+x^n)^n}{(1-x^{n+1})^{n+1}}$$

and let $\sum_{n=0}^{\infty}a(n)x^n$ be the corresponding power series. The statement of the following theorem was conjectured by Hanna in A325046.

Theorem 1. *For $n\in\mathbb{N}_0$, the coefficient $a(n)$ is odd if and only if $n=m(m+1)$, for some $m\in\mathbb{N}_0$.*

Proof. We have

$$\frac{1}{(1-x^{n+1})^{n+1}}=\sum_{k=0}^{\infty}\binom{n+k}{k}x^{(n+1)k},\qquad (1+x^n)^n=\sum_{i=0}^{n}\binom{n}{i}x^{ni}.$$

Thus,

$$A(x)=\sum_{n=0}^{\infty}\sum_{i=0}^{n}\sum_{k=0}^{\infty}\binom{n}{i}\binom{n+k}{k}x^{n(i+1)+(n+1)k}.$$

Therefore, for $N\in\mathbb{N}_0$, the parity of the coefficient $a(N)$ is equal to the parity of $\#P_N$, where

$$P_N=\left\{(n,i,k)\ :\ k,n\in\mathbb{N}_0, 0\leq i\leq n, N=n(i+1)+(n+1)k,\ \binom{n}{i}\binom{n+k}{k}\text{ odd}\right\}.$$

By Lucas' theorem modulo 2 [1], $\binom{n}{i}$ is odd if and only if in the binary expansions of i and n, the positions of ones in i are a subset of those in n. Furthermore, $\binom{n+k}{k}$ is odd if and only if adding n and k in binary produces no carries. Equivalently, k and n have disjoint positions of ones. Let

$$Q_N=\{(n,u)\in\mathbb{N}_0^2\ :\ N=nu+(n\vee u)\}.$$

We claim that $\#P_N=\#Q_N$. To this end, define a function $\Psi\colon P_N\to Q_N$ by $\Psi(n,i,k)=(n,i+k)$. To see that Ψ is well-defined, notice that since k

and n have disjoint positions of ones, $n + k = n \vee k$. Furthermore, i and k have disjoint positions of ones and therefore, their sum $u = i + k$ produces no carries. In addition, the positions of ones in i are a subset of the positions of ones in n. Thus, $n \vee k = n \vee (i + k) = n \vee u$. It follows that

$$N = n(i+1)+(n+1)k = n(i+k)+n+k = nu+(n\vee k) = nu+(n\vee u). \quad (1)$$

Thus, $(n, u) = (n, i+k) = \Psi(n, i, k) \in Q_N$. Conversely, we define a function $\Phi\colon Q_N \to P_N$ by $\Phi(n, u) = (n, i, k)$, where $i = u \wedge n$ and $k = u - (u \wedge n)$. Then $0 \leq i \leq n$ and $k \geq 0$. Furthermore, the positions of ones in i are a subset of the positions of ones in n, and k and n have disjoint positions of ones. In addition, i and k have disjoint positions of ones. It follows that (1) holds in this case as well, proving that $(n, i, k) = \Phi(n, u) \in P_N$. It may be verified that Φ and Ψ are inverse to one another and therefore bijections. It follows that it suffices to establish the parity of $\#Q_N$. To this end notice that the condition in Q_N is symmetric, i.e., $(n, u) \in Q_N \iff (u, n) \in Q_N$. Thus, elements (n, u) in Q_N with $n \neq u$ come in pairs and therefore do not affect the parity of $\#Q_N$. It remains to count elements in Q_N of the form (n, n). In this case, the condition in Q_N has the form $N = n^2 + (n \vee n) = n^2 + n = n(n + 1)$. Hence, $a(N)$ is odd if and only if $N = n(n + 1)$ for some $n \in \mathbb{N}_0$. $\square$

References

[1] E. Lucas, Theorie des fonctions numériques simplement périodiques, *Amer. J. Math.* **1** (1878), no. 2, 184–196 (Part 1); no. 3, 197–240 (Part 2); no. 4, 289–321 (Part 3).

[2] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, OEIS Foundation Inc., `https://oeis.org`.