A closed form formula for the determinant of a certain Toeplitz matrix

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For $n \in \mathbb{N}$ let M_n be the Toeplitz matrix whose first row is

$$(2n-1, n-1, n-2, \ldots, 1)$$

and whose first column is

$$(2n-1, 2n-2, 2n-3, \ldots, n).$$

Theorem 1. *Let $n \in \mathbb{N}$. Then*

$$\det(M_n) = n(n+1)^{n-1} + \frac{n-1}{4}((n-1)^{n-1} + (n+1)^{n-1}).$$

Proof. We have $M_1 = (1)$ and $\det(M_1) = 1$, agreeing with the asserted formula. Thus, assume that $n \geq 2$. First, we notice that

$$(M_n)_{ij} = \begin{cases} 2n-1-(i-j), & \text{if } i \geq j, \\ n-(j-i), & \text{if } i < j. \end{cases}$$

Let N_n be the matrix obtained from M_n by performing the row operations $R_i \leftarrow R_i - R_{i-1}, i = n, n-1, \ldots, 2$. It is easily verified that

$$(N_n)_{ij} = \begin{cases} 2n-1, & \text{if } i = j = 1, \\ n-j+1, & \text{if } i = 1 \text{ and } j \geq 2, \\ -1, & \text{if } i \geq 2 \text{ and } j \leq i-1, \\ n, & \text{if } i \geq 2 \text{ and } j = i, \\ 1, & \text{if } i \geq 2 \text{ and } j \geq i+1. \end{cases}$$

Now let K_n be the matrix obtained from N_n by performing the column operations $C_j \leftarrow C_j - C_{j+1}, i = 1, 2, \ldots, n-1$. It is easily verified that

$$(K_n)_{ij} = \begin{cases} n, & \text{if } i = j = 1, \\ 1, & \text{if } i = 1 \text{ and } 2 \leq j \leq n, \\ -(n+1), & \text{if } 2 \leq i \leq n \text{ and } j = i-1, \\ n-1, & \text{if } 2 \leq i \leq n-1 \text{ and } j = i, \\ 1, & \text{if } 2 \leq i \leq n-1 \text{ and } j = n, \\ n, & \text{if } i = j = n, \\ 0, & \text{otherwise.} \end{cases}$$

The matrix K_n may be written in block form as follows:

$$K_n = \begin{pmatrix} B_n & \mathbf{1} \\ \mathbf{c}^T & n \end{pmatrix},$$

where $\mathbf{1} = (1, \ldots, 1)^T$ is the all-ones vector of size $n-1$, $\mathbf{c}$ is the vector of size $n-1$ given by $\mathbf{c}^T = (0, \ldots, 0, -(n+1))$, and B_n is the square matrix of size $n-1$ given by

$$(B_n)_{ij} = \begin{cases} n, & \text{if } i = 1 \text{ and } j = 1, \\ 1, & \text{if } i = 1 \text{ and } 2 \leq j \leq n-1, \\ -(n+1), & \text{if } 2 \leq i \leq n-1 \text{ and } j = i-1, \\ n-1, & \text{if } 2 \leq i \leq n-1 \text{ and } j = i, \\ 0, & \text{otherwise.} \end{cases}$$

Expanding $\det(B_n)$ along the first row yields

$$\det(B_n) = \sum_{j=1}^{n-1} (B_n)_{1j} (C_n)_{1j},$$

where the $(C_n)_{1j}$s are the corresponding cofactors. It is not hard to see by induction on j that $C_{1j} = (n-1)^{n-1-j}(n+1)^{j-1}$. Thus,

$$\begin{aligned} \det(B) &= n(n-1)^{n-2} + \sum_{t=1}^{n-2} (n-1)^{n-2-t}(n+1)^t \\ &= \frac{(n-1)^{n-1} + (n+1)^{n-1}}{2}. \end{aligned}$$

In particular, B_n is invertible and by Schur's formula,

$$\det(K_n) = \det(B_n)(n - \mathbf{c}^T B_n^{-1}\mathbf{1}).$$

Let $x = B_n^{-1}\mathbf{1}$. To calculate x, we solve the system $B_n x = \mathbf{1}$, which may be written as

$$nx_1 + x_2 + \cdots + x_{n-1} = 1, \tag{1}$$

$$-(n+1)x_{i-1} + (n-1)x_i = 1, \qquad 2 \le i \le n-1. \tag{2}$$

From (2) we obtain the recurrence

$$x_i = \frac{1}{n-1} + \frac{n+1}{n-1}x_{i-1}, \qquad 2 \le i \le n-1. \tag{3}$$

Solving (3) gives

$$x_i = \left(\frac{n+1}{n-1}\right)^{i-1} x_1 + \frac{1}{2}\left(\left(\frac{n+1}{n-1}\right)^{i-1} - 1\right), \qquad 2 \le i \le n-1. \tag{4}$$

From (1) and (4) we obtain

$$x_1 = \frac{2n}{n - 1 + \dfrac{(n+1)^{n-1}}{(n-1)^{n-2}}} - \frac{1}{2}. \tag{5}$$

Taking $i = n-1$ in (4) and using (5), we have

$$x_{n-1} = \frac{2n(n+1)^{n-2}}{(n-1)^{n-1} + (n+1)^{n-1}} - \frac{1}{2}.$$

We conclude that

$$\begin{aligned}
\det(M_n) &= \det(N_n) = \det(K_n) = \det(B_n)(n - \mathbf{c}^T x) \\
&= \det(B_n)(n + (n+1)x_{n-1}) \\
&= \frac{(n-1)^{n-1} + (n+1)^{n-1}}{2}\left(n + (n+1)\left(\frac{2n(n+1)^{n-2}}{(n-1)^{n-1} + (n+1)^{n-1}} - \frac{1}{2}\right)\right). \\
&= n(n+1)^{n-1} + \frac{n-1}{4}((n-1)^{n-1} + (n+1)^{n-1}). \qquad \square
\end{aligned}$$

References

[1] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, OEIS Foundation Inc., `https://oeis.org`.