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On the Generating Functions of the Boas-Buck Sequences for the Inverse of Riordan and Sheffer Matrices

Wolfdieter L a n g [1]

Abstract

By a special application of the *Lagrange* series the generating functions for the Boas-Buck sequences for inverse Riordan or Sheffer matrices can be rewritten, and the coeffcients are obtained from the functions which determine the original Riordan or Sheffer matrix. The Boas-Buck sequences provide recurrences for each column sequence of such matrices. Several examples for the determination of the sequences for such inverse matrices are given.

1 Introduction and Summary

The *Boas-Buck* identity [1, 10] for ordinary and exponential lower triangular convolution matrices of the *Riordan* R or *Sheffer* S type, respectively,[11, 12, 5] imply a recurrence for these matrices involving the two *Boas-Buck* sequences $\alpha = \{\alpha_n\}_{n=0}^{\infty}$ and $\beta = \{\beta_n\}_{n=0}^{\infty}$. The ordinary generating functions (*o.g.f.* , for short) A and B of these sequences for both types of convolution matrices are given for $R = (G(x),\, x\,\widehat{F}(x))$ or $S = (g(x),\, x\,\widehat{f}(x))$ with formal power series (*f.p.s.* , for short) G, $\widehat{F}$ or g, $\widehat{f}$ are given as follows. Here $\widehat{F}$ or $\widehat{f}$, and usually also G or g, start with 1. For G or g the notation $\mathcal{G}$, and for $x\,\hat{F}$ or $x\,\hat{f}$ the notation $\mathcal{F}$ will be used, as well as $\widehat{\mathcal{F}}$ for $\mathcal{F} = x\,\widehat{\mathcal{F}}$. The notation of *Rainville* will be used.

$$A(x) \;:=\; \sum_{n=1}^{\infty} \alpha_n\, x^n \;=\; (\log\, \mathcal{G}(x))', \tag{1}$$

$$B(x) \;=\; \sum_{n=1}^{\infty} \beta_n\, x^n \;=\; (\log\, \widehat{\mathcal{F}}(x))'\,. \tag{2}$$

For completeness the recurrence formulae for the R and S matrix elements $R(n,\, m)$ and $S(n,\, m)$, as lower triangular matrices, vanishing for $n < m$, are also given.

$$R(n,m) \;=\; \frac{1}{n-m} \sum_{k=m}^{n-1} (\alpha_{n-1-k} \,+\, m\,\beta_{n-1-k})\, R(k,\, m)\,, \quad \text{for} \;\; n \in \mathbb{N}, \;\text{and}\; m \,=\, 0,\, 1,\, ...,\, n-1\,, \tag{3}$$

$$S(n,m) \;=\; \frac{n!}{n-m} \sum_{k=m}^{n-1} \frac{1}{k!}\, (\alpha_{n-1-k} \,+\, m\,\beta_{n-1-k})\, S(k,\, m)\,, \quad \text{for} \;\; n \in \mathbb{N}, \;\text{and}\; m \,=\, 0,\, 1,\, ...,\, n-1\,, \tag{4}$$

$$\tag{5}$$

with the diagonal elements $R(n,\, n)$ or $S(n,\, n)$ as inputs.

These matrices form a group, the *Riordan* or *Sheffer* group. These square matrices are infinite dimensional but one can consider any finite dimension N for practical purposes. Subgroups of special interest are the so-called associated groups with $\mathcal{G} = 1$, and the *Bell*, resp. *Narumi*, groups with $\widehat{\mathcal{F}} = \mathcal{G}$.
In this note we are interested in inverse *Riordan* R^{-1} and *Sheffer* matrices S^{-1}.These inverse matrices are denoted by $R^{-1} = \left(\frac{1}{G \circ F^{[-1]}},\, F^{[-1]}\right)$ or $S^{-1} = \left(\frac{1}{g \circ f^{[-1]}},\, f^{[-1]}\right)$, were the compositional inverse of the *f.p.s.* F and f are denoted by $F^{[-1]}$ and $f^{[-1]}$, respectively. The composition symbol $\circ$ means that $(g \circ f)(x) := g(f(x))$. Thus $f \circ f^{[-1]} = id = f^{[-1]} \circ f$.

[1] wolfdieter.lang@partner.kit.edu, http://www.itp.kit.edu/~wl

In the *Sheffer* case the subgroup of the associated matrices $J = (1, f)$ is also called *Jabotinsky* subgroup, It appears more often than the *Narumi* case. For this case the inverse matrix is $J^{-1} = (1, f^{[-1]}(y)$. These two cases are closely related: for $N = (g(x), x\,g(x)) = \left(\frac{f(x)}{x}, f(x)\right)$ and $J = (1, f(x))$ the matrix elements are related by $N(n, m) = \frac{m+1}{n+1}\, J(n+1, m+1)$, and the row polynomials are $Npol(n, x) = \frac{1}{n+1}\, Jpol'(n+1, x)$. This follows from the corresponding *e.g.f.* s of the column sequences, and the definition of the row polynomials.

The generating functions for the Boas-Buck sequences (we use the notations $\overline{\alpha}$, $\overline{\beta}, \overline{A}$, and $\overline{B}$) for these inverse matrices, using, as above, $\mathcal{G}$ for G or g, and $\mathcal{F}^{[-1]}$ for $F^{[-1]}$ or $f^{[-1]}$ are then

$$\overline{A}(y) \;=\; \sum_{n=1}^{\infty} \overline{\alpha}_n\, y^n \;=\; -(\log \mathcal{G}(\mathcal{F}^{[-1]}(y)))', \tag{6}$$

$$\overline{B}(y) \;=\; \sum_{n=1}^{\infty} \overline{\beta}_n\, y^n \;=\; (\log \mathcal{F}^{[-1]}(y))' - \frac{1}{y}\,. \tag{7}$$

A special application of the *Lagrange* series (see *e.g.*, [3, 13]) can now be used to determine the ordinary generating functions $\overline{A}$ and $\overline{B}$ for both cases R^{-1} and S^{-1}.

Theorem 1. *Lagrange theorem and inversion* [3, 13]

a) *For $\widetilde{H}(x) = H(y(x))$ with implicit $y = y(x) = a + x\,\varphi(y)$ (here as f.p.s.) one has*

$$\widetilde{H}(x) \;=\; H(a) + \sum_{n=1}^{\infty} \frac{x^n}{n!} \frac{d^{n-1}}{da^{n-1}} \left[\varphi^n(a)\, H'(a)\right]. \tag{8}$$

b) *With $a = 0$, $y = y(x) = x\,\psi(x)$, and the compositional inverse $x = y^{[-1]} = x(y)$ it follows that*

$$\begin{aligned}\tilde{H}(y) \;=\; H(x(y)) \;&=\; H(0) + \sum_{n=1}^{\infty} \frac{y^n}{n!} \frac{d^{n-1}}{da^{n-1}} \left[\left(\frac{1}{\psi(a)}\right)^n H'(a)\right]\Bigg|_{a=0} \\ &=\; H(0) + \sum_{n=1}^{\infty} \frac{y^n}{n!} (n-1)!\,[a^{n-1})] \left[\left(\frac{1}{\psi(a)}\right)^n H'(a)\right]\end{aligned} \tag{9}$$

where $[a^n]\, h(a)$ picks the coefficient of a^n of a f.p.s. $h = h(a)$.

Applying this theorem, part **b)** to $y = y(x) = x\,\psi(x)$ with the *Riordan* or *Sheffer* function $\psi = \widehat{\mathcal{F}}$, and choosing $\frac{d}{dt}\, H(t) = \psi(t)$, we obtain with the compositional inverse $y^{[-1]} = \mathcal{F}^{[-1]}$ of $y = y(x) = \mathcal{F}(x) = x\,\widehat{\mathcal{F}}(x)$, and after differentiation, the following proposition for the *o.g.f.* $\mathcal{T}$ of the sequence $\{t_n\}_{n=0}^{\infty}$.

Proposition 2. *The o.g.f. $T(y) = \sum_{n=0}^{\infty} t_n\, y^n$ with $t_n := \frac{1}{n!} \frac{d^n}{dt^n} \frac{1}{(\widehat{\mathcal{F}}(t))^n}\Big|_{t=0}$ obeys*

$$T(y) \;=\; \widehat{\mathcal{F}}(\mathcal{F}^{[-1]}(y))\,(\mathcal{F}^{[-1]}(y))' \;=\; \frac{\widehat{\mathcal{F}}(x)}{\mathcal{F}'(x)}\bigg|_{x=\mathcal{F}^{[-1]}(y)} \;=\; \frac{y}{\mathcal{F}^{[-1]}(y)} \frac{1}{\mathcal{F}'(x)}\bigg|_{x=\mathcal{F}^{[-1]}(y)}. \tag{10}$$

Proof. With the choice $\frac{d}{dt}\, H(t) = \widehat{\mathcal{F}}(t)$ we have from the first version of eq. 9

$$\tilde{H}(y) \;=\; H(F^{[-1]}(y)) \;=\; H(0) + \sum_{n=1}^{\infty} \frac{y^n}{n!} \left[\frac{d^{n-1}}{dt^{n-1}} \left(\frac{1}{\widehat{\mathcal{F}}(t)}\right)^{n-1}\right]_{t=0}. \tag{11}$$

The sum becomes $\displaystyle\sum_{n=0}^{\infty} \frac{y^{n+1}}{n+1} \left[\frac{1}{n!} \frac{d^n}{dt^n} \left(\frac{1}{\widehat{\mathcal{F}}(t)}\right)^n\right]_{t=0} = \int dy \sum_{n=0}^{\infty} y^n\, t_n$, without worrying about interchanging the sum with the integral for a formal power series. Differentiation on both sides leads, with the chain rule for $\dfrac{d}{dy} H(F^{[-1]}(y))$, and for the derivative of the compositional inverse, to

$$T(y) \;=\; \widehat{\mathcal{F}}(\mathcal{F}^{[-1]}(y))\, \frac{d}{dy} \mathcal{F}^{[-1]}(y) \;=\; \frac{\widehat{\mathcal{F}}(x)}{\mathcal{F}'(x)}\bigg|_{x=\mathcal{F}^{[-1]}(y)} \;=\; \frac{y}{\mathcal{F}^{[-1]}(y)} \frac{1}{\mathcal{F}'(x)}\bigg|_{x=\mathcal{F}^{[-1]}(y)}. \tag{12}$$

$\square$

The *o.g.f.* $T(y)$ coincides up to the offset with *o.g.f.* for $\overline{B}(y)$.

Corollary 3.

$$\overline{B}(y) = \frac{1}{y}\,(T(y)-1) = \frac{1}{\mathcal{F}^{[-1]}(y)}\,\frac{1}{\mathcal{F}'(x)}\bigg|_{x=\mathcal{F}^{[-1]}(y)} - \frac{1}{y}\,, \tag{13}$$

$$\overline{\beta}_n = t_{n+1},\quad \text{for } \mathrm{n} \in \mathbb{N}_0, \quad \text{and}\quad \overline{\beta}_{-1} = 1\,. \tag{14}$$

The other *Boas-Buck o.g.f.* $\overline{A}$ of eq. 6 is first rewritten, in order to choose the function $H(t)$ appropriately. The chain rule for $h(y) := \mathcal{G}(\mathcal{F}^{[-1]}(y))$, and for the derivative of the compositional inverse, leads to

$$\overline{A}(y) = -\frac{h'(y)}{h(y)} = -\frac{(\log \mathcal{G}(x))'}{\mathcal{F}'(x)}\bigg|_{x=\mathcal{F}^{-1}(y)}\,. \tag{15}$$

Because $\frac{d}{dy}\,H(\mathcal{F}^{[-1]}(y)) = \frac{H'(x)}{\mathcal{F}'(x)}\Big|_{x=\mathcal{F}^{[-1]}(y)}$ the choice in eq. 9 is now $H(t) = -\log \mathcal{G}(t)$, with $H(0) = 0$ from the assumed $\mathcal{G}(0) = 1$.

After these preliminaries one finds, with $y = y(x) = x\,\widehat{\mathcal{F}}(x)$ as above, after differentiating eg. 9

$$\frac{-(\log \mathcal{G}(x))'}{\mathcal{F}'(x)}\bigg|_{x=\mathcal{F}^{[-1]}(y)} = \sum_{n=0}^{\infty} y^n \left[\frac{1}{n!}\,\frac{d^n}{dt^n}\,\frac{-(\log \mathcal{G}(t))'}{(\widehat{\mathcal{F}}(t))^{n+1}}\right]_{t=0}\,. \tag{16}$$

This implies the following *Proposition.*

Proposition 4. *The o.g.f.* $S(y) = \sum_{n=0}^{\infty} s_n\, y^n$ *with* $s_n := \frac{1}{n!}\frac{d^n}{dt^n}\frac{-(\log \mathcal{G}(t))'}{(\widehat{\mathcal{F}}(t))^{n+1}}\Big|_{t=0}$ *obeys*

$$S(y) = \frac{-(\log \mathcal{G}(x))'}{\mathcal{F}'(x)}\bigg|_{x=\mathcal{F}^{[-1]}(y)}\,. \tag{17}$$

Corollary 5. *The o.g.f.* $S(y)$ *coincides with the o.g.f.* $\overline{A}(y)$, *and* $\overline{\alpha}_n = s_n$ *for* $n \in \mathbb{N}_0$.

For the *Bell* or *Narumi* sub-groups, with $\mathcal{G} = \widehat{\mathcal{F}}$ the generating functions for the *Boas-Buck* sequences simplify.

Corollary 6. Boas-Buck sequences for inverse Bell- or Narumi-type matrices

$$\overline{A}(y) = \overline{B}(y) = -\frac{(\log \widehat{\mathcal{F}}(x))'}{\mathcal{F}'(x)}\bigg|_{x=\mathcal{F}^{-1}(y)} = \frac{1}{\mathcal{F}^{[-1]}(y)}\,\frac{1}{\mathcal{F}'(x)}\bigg|_{x=\mathcal{F}^{[-1]}(y)} - \frac{1}{y}\,, \tag{18}$$

$$\overline{\alpha}_n = \overline{\beta}_n = \frac{1}{(n+1)!}\,\frac{d^{n+1}}{dt^{n+1}}\,\frac{1}{(\widehat{\mathcal{F}}(t))^{n+1}}\bigg|_{t=0}\,. \tag{19}$$

Proof. For the $\mathcal{G}$ function of the inverse matrix, $\mathcal{G}_{\rangle\backslash\sqsubseteq}$, of a *Bell* or *Narumi* matrix $(\mathcal{G},\,\mathcal{F})$ one has $\mathcal{G}_{inv}(y) = \frac{1}{\mathcal{G}(\mathcal{F}^{[-1]}(y))} = \frac{\mathcal{F}^{[-1]}(y)}{y}$ because the $\mathcal{F}$ function of the inverse matrix is $\mathcal{F}^{[-1]}$. In this sub-group $\overline{A}(y) = \overline{B}(y)$ from the defining eqs. 6 and 7. To see that indeed $\overline{\beta}_n = \overline{\alpha}_n$ one computes the first inner derivative and splits of the logarithmic derivative of $\mathcal{F}$. ◻

Corollary 7. Boas-Buck sequences for inverse Jabotinsky-type matrices

$$\overline{A}(y) = 0,\quad \overline{B}(y) = \frac{1}{f^{[-1]}(y)}\,\frac{1}{f'(x)}\bigg|_{x=f^{[-1]}(y)} - \frac{1}{y}\,, \tag{20}$$

$$\overline{\alpha}_n = 0,\quad \overline{\beta}_0 = 0,\quad \text{and}\quad \overline{\beta}_{n-1} = \frac{1}{n!}\,\frac{d^n}{dt^n}\,\frac{1}{(\widehat{f}(t))^n}\bigg|_{t=0},\quad \text{for}\ \ n \in \mathbb{N}\,. \tag{21}$$

2 Examples

A) Riordan case

1) *Bell*-type $R = \left(\frac{1}{1-x}, \frac{x}{1-x}\right)$ = A007318 (*Pascal*) matrix with its inverse $R^{-1} = \left(\frac{1}{1+t}, \frac{t}{1+t}\right)$ with elements $R^{-1}(n,m) = (-1)^{n-m}\, R(n,m) = (-1)^{n-m}\binom{n}{m}$.

$$A(x) \;=\; B(x) \,=\, \frac{1}{1\,-\,x} \,=\, \hat{F}(x),\;\; \overline{A}(y) \,=\, \overline{B}(y) \,=\, \frac{-1}{1\,+\,y}\,, \tag{22}$$

$$\alpha_n \;=\; \beta_n \,=\, 1,\;\; \overline{\alpha}_n \,=\, \overline{\beta}_n \,=\, (-1)^{n+1},\;\; \text{for}\;\; n\,\in\,\mathbb{N}_0\,. \tag{23}$$

In this case the result from the *Lagrange* series eq. 19 is $[t^{n+1}]\,(1\,+\,t)^{n+1}$, the coefficient of t^{n+1} which is $(-1)^{n+1}$. The *Boas-Buck* column recurrence reduces to the binomial identity derived from the *Boas-Buck* column recurrence for the *Pascal* columns

$$\binom{n}{m} \,=\, \sum_{k=m}^{n-1}\binom{k}{m},\;\; \text{for}\;\; m\,\in\,\mathbb{N}_0,\;\; \text{and}\;\; n\,\geq\,m\,+\,1\,. \tag{24}$$

This identity is a rewritten form of the one found in [4] on p. 161.

2) $R = \left(\frac{1}{(1-x)^2}, \frac{x}{1-x}\right)$ = A135278, with the inverse Riordan matrix $R^{-1} = \left(\frac{1}{(1+x)^2}, \frac{x}{1+x}\right)$.

$$A(x) \;=\; \frac{2}{1\,-\,x},\;\; B(x) \;=\; \frac{1}{1\,-\,x} \,=\, \hat{F}(x),\;\; \overline{A}(y) \;=\; -2\,\frac{1}{1\,+\,y}.\;\; \overline{B}(y) \;=\; -\frac{1}{1\,+\,y}\,, \tag{25}$$

$$\alpha_n \;=\; 2,\;\; \beta_n \,=\, 1,\;\; \overline{\alpha}_n \,=\, 2\,(-1)^{n+1},\;\; \overline{\beta}_n \,=\, (-1)^{n+1},\;\; \text{for}\;\; n\,\in\,\mathbb{N}_0\,. \tag{26}$$

The result from the *Lagrange* series for $\overline{\alpha}_n$, *Proposition* 4 and *Corollary* 5, is $[t^n]\,(-2\,(1\,+\,t)^n)$, which is $-2\,(-1)^n$; and for $\overline{\beta}_{n-1}$, from *Proposition* 2 and *Corollary* 3, it is $[t^n]\,((1\,+\,t)^n)$, which is $(-1)^n$.

3) *Bell*-type $R = \left(\frac{1}{1-x^2-x^3}, \frac{x}{1-x^2-x^3}\right)$ = A104578 (*Padovan*), and $R^{-1} = (h(y),\, y\,h(y))$, with $h(y) := \frac{F^{[-1]}(y)}{y}$, where $F(x) = \frac{x}{1-x^2-x^3}$. The expansion of f is given in A319201 = $\{1,\,0,\,-1,\,-1,\,2,\,5,\,-2,\,-21,\,-14,\,72,\,138,\,...\}$.

$$A(x) \;=\; B(x) \,=\, \frac{x\,(2\,+\,3\,x)}{1\,-\,x^2\,-\,x^3},\; \overline{A}(y) \,=\, \overline{B}(y) \,=\, (1/(1/h(y) + y^2\,h(y) + 2\,y^3\,h(y)^2) - 1)/y, \tag{27}$$

$$\alpha_n \;=\; \beta_n \,=\, \text{A001608}(n+1),\;\; \overline{\alpha}_n \,=\, \overline{\beta}_n \,=\, \text{A319204}(n),\;\; \text{for}\;\; n\,\in\,\mathbb{N}_0\,. \tag{28}$$

In the formula for $\overline{A}(y) = \overline{B}(y) = (\log(h(y)))'$ from eq. 18 the following identity for $h(x)$, implied by the equation for $F^{[-1]}$, can be used repeatedly to reduce powers of f to h^2, h, 1.

$$h^3(y) \,=\, \frac{1}{y^3}\,\left(-(y\,h(y))^2\,-\,h(y)\,+\,1\right)\,. \tag{29}$$

The result for $\overline{\alpha}_n = \overline{\beta}_n$ from the *Lagrange* approach series $s_n = \overline{\alpha}_n$ of *Proposition* 4 is $[t^n]\,(-t\,(2\,+\,3\,t)\,(1\,-\,t^2\,-\,t^3)^n)$, the coefficient of t^n, which is A319204(n) = $\{0,\,-2,\,-3,\,6,\,20,\,-5,\,-105,\,-98,\,...\}$. However, the result given in eq. 19 for $\overline{\beta}_{n-1} = t_n$ from *Proposition* 2, is the simpler, *i.e.*, $[t^n]\,(1\,-\,t^2\,-\,t^3)^n$, for $n \in \mathbb{N}$. This can be computed from the multinomial formula for $(x1\,+\,x2\,+\,x3)^n$, setting $x1 = 1$, $x2 = -t^2$ and $x3 = -t^3$. This leads to

$$\overline{\beta}_{n-1} \,=\, \sum_{2\,e2\,+\,3\,e3\,=\,n} (-1)^{e2+e3}\,\frac{n!}{(n-(e2+e3))!\,e2!\,e3!}\,, \tag{30}$$

with nonnegative integers $e2$ and $e3$. The solutions for the pairs $(e2,\,e3)$ are given in table A321201, and the corresponding (unsigned) multinomials are found in A321203. The parities of $e2\,+\,e3$ have to be taken into account in summing the entries of row n. *E.g.*, , for $n = 6$ one has for $\overline{\beta}_5 = -5$ to sum $+15\,+\,(-20) = -5$ because the two $e2$, $e3$ pairs are $(0,\,2)$ and $(3,\,0)$.

B) Sheffer case

1) (*Jabotinsky*-type) $S2 \;=\; (1, exp(x) - 1) \;=$ A048993 (*Stirling2*), and $S2^{-1} \;=\; S1 \;=\; (1, \log(1+y)) \;=$ A048994 (*Stirling1*)

$$\begin{aligned} A(x) &= 0,\;\; B(x) \,=\, \frac{1 - e^{x} + x\,e^{x}}{x\,(e^{x} - 1)},\;\; \overline{A}(y) \,=\, 0,\, \overline{B}(y) \,=\, \frac{1}{\log(1+y)\,(1+y)} - \frac{1}{y}, & (31)\\ \alpha_n &= 0,\;\; \beta_n \,=\, \frac{(-1)^{n+1}\,\text{A060054}(n+1)}{\text{A227830}(n+1)} \,=\, \frac{(-1)^{n+1}\,Bernoulli(n+1)}{(n+1)!} \text{ for } n \in \mathbb{N}_0, & (32)\\ \overline{\alpha}_n &= 0,\;\; \overline{\beta}_n \,=\, \frac{(-1)^{n+1}\text{A002208}(n+1)}{\text{A002209}(n+1)},\text{ for } n \in \mathbb{N}_0. & (33) \end{aligned}$$

The formula $\overline{\beta}_{n-1} = t_n$, for $n \in \mathbb{N}$, with t_n from *Proposition* 2 of the *Lagrange* series approach, needs $(2\,n)$-fold application of *Hôpital*'s rule.

2) $S[3,1] \;=\; (e^{x}, e^{3\,x} - 1) \;=$ A282629, and the inverse matrix $S1[3,1] \;=\; \left(\frac{1}{1+y)^{1/3}}, \frac{1}{3}\log(1+x)\right)$ with $S1[3,1](n,k) \;=\; (-1)^{n-k}$ A286718$(n,k)/3^{n}$.

$$\begin{aligned} A(x) &= 1,\;\; B(x) \,=\, \frac{3\,x\,e^{3\,x} - e^{3\,x} + 1}{x\,(e^{3\,x} - 1)},\;\; \overline{A}(y) \,=\, -\frac{1}{3\,(1+x)},\, \overline{B}(y) \,=\, \frac{y - (1+y)\log(1+y)}{y\,(1+y)\log(1+y)}, & (34)\\ \alpha_0 &= 1,\; \alpha_n \,=\, 0, \text{ for } n \in \mathbb{N},\;\; \beta_n \,=\, \frac{3\cdot\text{A321329}(n)}{\text{A321330}(n)}, = (-3)^{n+1}\,\frac{Bernoulli(n+1)}{(n+1)!} \text{ for } n \in \mathbb{N}_0, & (35)\\ \overline{\alpha}_n &= \frac{(-1)^{(n+1)}}{3},\;\; \overline{\beta}_n \,=\, \frac{(-1)^{n+1}\text{A002208}(n+1)}{\text{A002209}(n+1)},\text{ for } n \in \mathbb{N}_0. & (36) \end{aligned}$$

Note that the *o.g.f.* $\overline{B}(y)$ coincides with the one of *Example* **1)**.

The formula $\overline{\alpha}_n \;=\; s_n \;=\; \frac{1}{n!}\frac{d^n}{dt^n}\left(-\left(\frac{t}{e^{3\,t} - 1}\right)^{n+1}\right)$ from *Proposition* 4 needs $(2\,n + 1)$-fold application of *Hôpital*'s rule. *E.g.*, for $n = 1$ one obtains $\overline{\alpha}_1 \;=\; \frac{54}{162} \;=\; \frac{1}{3}$.

The formula $\overline{\beta}_{n-1} \;=\; t_n \;=\; \frac{1}{n!}\frac{d^n}{dt^n}\left(\frac{t}{e^{3\,t} - 1}\right)^{n}$, for $n \geq 1$, from *Proposition* 2, needs $(2\,n)$-fold application of *Hôpital*'s rule. *E.g.*, for $n = 2$ one obtains, after 4-fold application of *Hôpital*'s rule, $\overline{\beta}_1 \;=\; \frac{810}{1944} \;=\; \frac{5}{12}$.

3) (*Narumi* type) $N \;=\; \left(\frac{\log(1+x)}{x}, \log(1+x)\right) \;=\; \frac{1}{n+1}$A028421$(n,\,m)\,(-1)^{n-m}$ (*Narumi* $a \;=\; -1$), and $N^{-1} \;=\; \left(\frac{e^{y}-1}{y}, e^{y} - 1\right)$, $N^{-1}(n,\,m) \;=\; \frac{1}{n+1}$ A321331$(n,\,m) \;=\; \frac{m+1}{n+1}\,S2(n+1,\,m+1)$ with $S2 =$A048993, (*Stirling2*).

$$\begin{aligned} A(x) &= B(x) \,=\, \frac{1}{(1+x)\,ln(1+x)} - \frac{1}{x},\;\; \overline{A}(y) \,=\, \overline{B}(y) \,=\, \frac{e^{y}}{e^{y} - 1} - \frac{1}{y}, & (37)\\ \alpha &= \beta_n \,=\, \frac{(-1)^{n+1}\text{A002208}(n+1)}{\text{A002209}(n+1)},\text{ for } n \in \mathbb{N}_0, & (38)\\ \overline{\alpha}_0 &= \overline{\beta}_0 \,=\, \frac{1}{2},\, \overline{\alpha}_n \,=\, \overline{\beta}_n \,=\, \frac{(-1)^{n+1}\text{A060054}(n+1)}{\text{A227830}(n+1)} \,=\, \frac{(-1)^{n+1}\,Bernoulli(n+1)}{(n+1)!} \text{ for } n \in \mathbb{N}_0, & (39)\\ &. & (40) \end{aligned}$$

$A(x) \;=\; B(x)$ coincides with $\overline{B}(x)$ of *Example* **1)**, and $\overline{A}(x) \;=\; \overline{B}(x)$ coincides with $B(x)$ of *Example* **1)**.

The formula $\overline{\beta}_{n-1} = t_n$, for $n \in \mathbb{N}$, with t_n from *Proposition* 2 of the *Lagrange* series approach, needs $(2\,n)$-fold application of *Hôpital*'s rule.

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