THE SURFER AND THE HUT: A POLYGON DISSECTION

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Abstract. A surfer stranded on an island with a shoreline shaped as a regular n-sided polygon might consider the best spot of his hut to be where the mean distance to the n beaches is a minimum. For the regular triangle, Viviani's theorem states that this mean distance does not actually depend on the placement of the hut.

For $n \geq 5$, however, there exist points on the island where the location of the hut does not minimize the mean distance to the n beaches. Minimizing points for the hut can be found by constructing perpendiculars to the sides at the vertices. This introduces a tile dissection of the island's surface. We prove the formulas which count how many tiles the n-gon island is dissected into by these perpendiculars, which turn out to be second-order polynomials in n.

1. Viviani's Theorem

The generalized form of Viviani's theorem states that, in a regular $n-$gon, the sum of the distances to the sides from any interior point is equal to n times the apothem [1, 10, 11, 2, 3].

For instance, in the equilateral triangle ABC in Figure 1, HP_1 is the distance from interior point H to side AB, HP_2 is the distance from H to side BC, and HP_3 is the distance from H to side CA. The sum $HP_1+HP_2+HP_3$ is equal to 3 times the apothem, which also is equal to the height of the triangle.

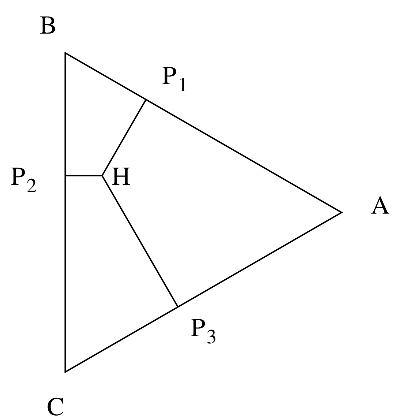

Figure 1. Viviani's theorem in the equilateral triangle ABC.

The sum is independent of the location of H, according to Viviani's theorem.

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Pickover relates [6, 4, 9]:

> Some teachers have placed the problem in a real-world setting by casting it in the context of a surfer stranded on an island in the shape of an equilateral triangle. The surfer wants to build a hut where the sum of the distances to the sides is a minimum, because she surfs on each of the three beaches an equal amount of time. Students are intrigued to learn that the placement of the hut doesn't matter.

This surfer-and-the-hut context also works on a square island, where the surfer can place the hut anywhere on the square island, and the sum of the distances to the 4 beaches remains a minimum.

2. Higher Polygons

However, in the case of a regular pentagonal island, there are points on the island where the surfer cannot place the hut, and still maintain the minimum distance. Consider the regular pentagon ABCDE in Figure 2, and point H. For this location of the Hut H, and for Viviani's theorem to hold, the surfer would have to travel to the extensions of sides AB and CD. This is impractical in reality.

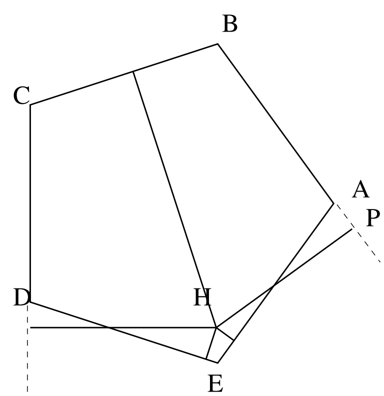

Figure 2. Hut at point H in the equilateral pentagon ABCDE.

It could be argued that by traveling from H to P, the B surfer would not be surfing in the waters of beach AB, but rather in the waters of beach AE. To travel the shortest distance to beach AB, the surfer would have to travel to point A, which is longer than HP.

In the real-world context of the surfer and the hut, and for all regular n-gon islands $n \geq 5$, there exist points on the island where the location of the hut does not minimize the sum of the distances to the beaches. The location of the viable points for the hut can be found by constructing the perpendiculars to the sides at the vertices. For instance, in the case of the regular 5-gon island, the viable points for the hut lie on the regular decagon (and its interior) created by constructing the perpendiculars, as shown in Figure 3.

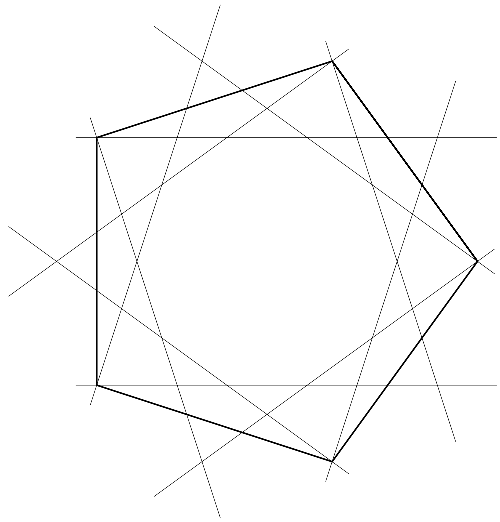

FIGURE 3. Perpendiculars in the regular 5-gon. The common area where all viable rectangular regions defined by the perpendiculars intersect is a 10-gon in the center.

In the context of the surfer and the hut, and for all regular n-gon islands $n \geq 3$, the feasible region for the hut is defined by a regular polygon with

$$\begin{cases} n, & n \leq 4 \quad \text{or } n \text{ even}, \\ 2n, & n \geq 5 \quad \text{and } n \text{ odd}, \end{cases} \tag{1}$$

sides.

Remark 1. *The symmetry group of the geometry is the dihedral group D_{2n} of order $2n$, where the two generators of the group that leave the geometry invariant are (i) rotating the polygon and perpendiculars by the angle $2\pi/n$ around the center and (ii) flipping them around a line that contains the center and an edge of the n-gon. From that argument the center polygon of the feasable region must be compatible with that symmetry and can be any k-gon with a symmetry which has D_{2n} as a subgroup. The subgroup structure of dihedral groups then enforces that the symmetry group of the center polygon is D_{2k} and that k is a multiple of n. The fact that there are only n-gons and $2n$-gons in the center is merely reflecting that each edge of the polygon in the center must be a section of a perpendicular, and that there are overall at most $2n$ perpendiculars, pairwise equal if n is even.*

For instance, in the case of the regular 6-gon island, the viable points for the hut lie on the regular hexagon (and its interior) created by constructing the perpendiculars, as shown in Figure 4.

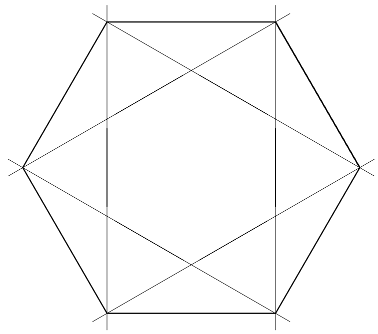

Figure 4. Perpendiculars in the regular 6-gon. The viable region is a 6-gon in the center.

3. Counting Tiles

An interesting byproduct of the conjectured method for determining the feasible region (i.e., constructing the perpendiculars to the sides at the vertices) is that this construction dissects the regular n-gon in a predictable manner. Taking advantage of the n-fold symmetry axis in the center (the cyclic subgroup of D_{2n}), one can economically enumerate triangular wedges with pen and paper that are only $1/n$-th of the interior of the n-gon from 1 to $f(n)$, and derive the full count by copying these slices n times around, and adding the single cell in the center:

$$F(n) = 1 + nf(n), \quad n \geq 3. \tag{2}$$

The 5-gon and the 6-gon enumerated with these symmetry-reduced labels are shown in Figure 5. One wedge is shaded in each case. Where a tile is partially outside a wedge, it contributes 1/2 to $f(n)$. [Because this happens always twice like for tile number 6 in the 5-gon and tile number 2 in the 6-gon, $f(n)$ remains an integer.]

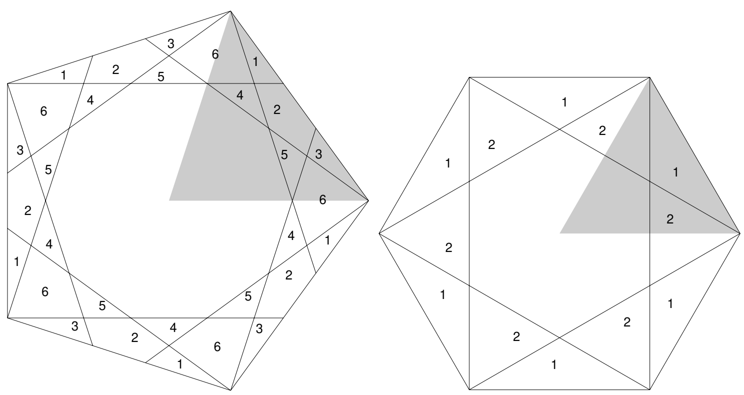

Figure 5. The 5-gon, enumerated with $f(5) = 6$ for a total of $F(5) = 1+5\times6 = 31$ tiles, and the 6-gon enumerated with $f(6) = 2$ for a total of $F(6) = 1 + 6 \times 2 = 13$ tiles.

Pictures for the 7, 8, 9, 10, 11, and 12-gon are gathered in Figure 6.

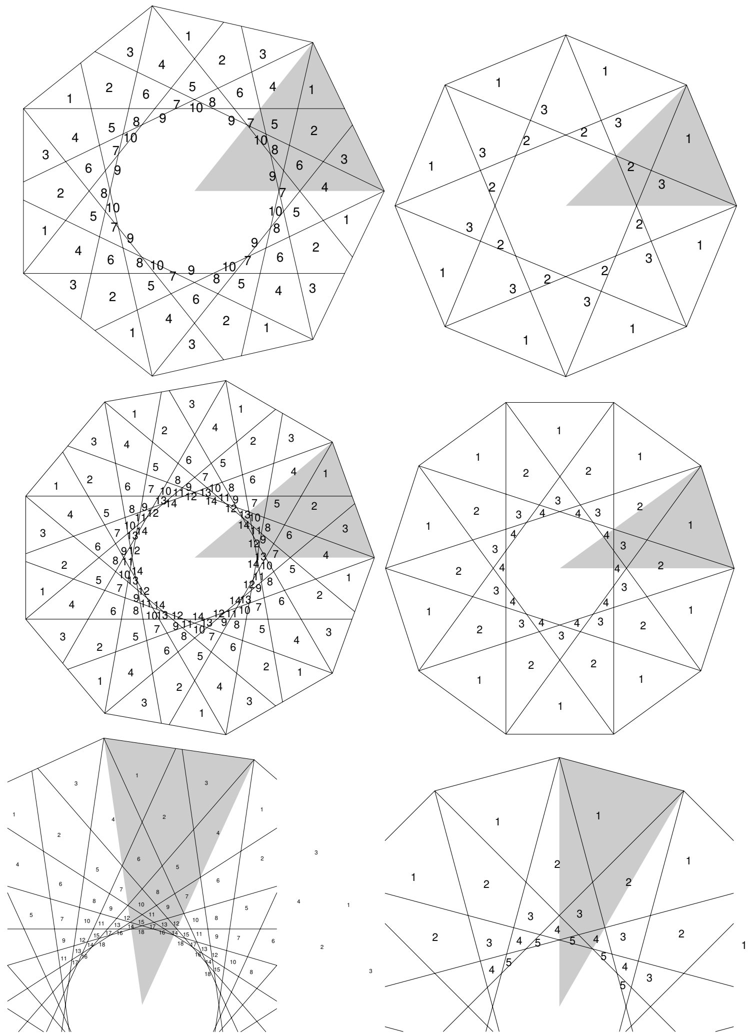

Figure 6. Perpendiculars in the 7-gon to 12-gon. $f(7) = 10$, $f(8) = 3$, $f(9) = 14$, $f(10) = 4$, $f(11) = 18$ and $f(12) = 5$.

Looking at Figure 6, enumerations of the 7-gon yields a total of $1 + 7 \times 10 = 71$ pieces, the 8-gon yields $1 + 8 \times 3 = 25$ pieces, and the 9-gon yields $1 + 9 \times 14 = 127$ pieces. Given a regular n-gon, with $n \geq 5$, and dissecting the n-gon by constructing perpendiculars to the sides at the vertices, yields the sequence [5, A320431]

$$F(n) = 1, 1, 31, 13, 71, 25, 127, \ldots = \begin{cases} 1, & n = 3, 4; \\ 1 + 2n(n-2), & n \geq 5 \text{ odd}; \\ 1 + n(\frac{n}{2} - 1), & n \geq 5 \text{ even}. \end{cases} \tag{3}$$

This formula is derived in Section 4 and compatible with the form (2).

4. Euler's Formula

Equation (3) can be derived from Euler's formula that associates the number of vertices V, the number of edges E, and the number of faces F in a graph:

$$1 + E = F + V. \tag{4}$$

Remark 2. *We set the term on the left-hand side to 1, not 2, because we do not count the face represented by the area outside the n-gon.*

4.1. Odd n.

If n is odd, each of the edges of the n-gon is cut into 3 pieces by the two perpendiculars of the almost-opposite edge. It is not 2 pieces because these perpendiculars never meet in the middle as set out in Section B. So $3n$ is the number of edges along the n-gon and $3n$ the number of vertices of the n-gon.

Each of the $2n$ perpendiculars is also intersected by $2n - 6$ other perpendiculars inside the n-gon. From the total of $2n$ perpendiculars these are the 6 that do *not* cross a perpendicular starting at vertex j inside the n-gon (but on the edges or outside), sketched in Figure 7:

(1) The perpendicular itself;
(2) The perpendicular which starts at the same vertex j, perpendicular to the adjacent side;
(3) The perpendicular that runs parallel to it. This starts at vertex $j + 1$ or vertex $j - 1$.
(4) The perpendicular that hits the same opposite edge than this one, which starts at vertex $j + 2$ or vertex $j - 2$. (Section B shows that these two cross *outside* the n-gon.)
(5) One of the 4 perpendiculars starting at the edge of the n-gon that is hit by this perpendicular
(6) One of the 4 perpendiculars starting at the edge opposite to the vertex

That splits each of the perpendiculars into $2n - 5$ pieces inside the n-gon, a total of $2n(2n - 5)$ edges of polygons inside the n-gon. Adding the $3n$ pieces along the perimeter we count $E = 3n + 2n(2n - 5) = 4n^2 - 7n$ edges. The $2n - 6$ crossings counted above on each diagonal define $2n(2n - 6)/2$ vertices inside the n-gon; we divide by 2 here because there are $2n$ perpendiculars but each crossing would be counted separately (twice) for each perpendicular. Adding the $3n$ vertices along the perimeter considered above, $V = 3n + 2n(2n - 6)/2 = 2n^2 - 3n$. Formula (4) yields $F = 1 + (4n^2 - 7n) - (2n^2 - 3n) = 2n^2 - 4n + 1$ as claimed in (3).

4.2. Even n.

If n is even, the perpendiculars run between vertices of the n-gon, so there are n perpendiculars and the edges of the n-gon contribute n to V and contribute n to E in Euler's formula. The perpendicular starting at vertex $\vec{v}_j$

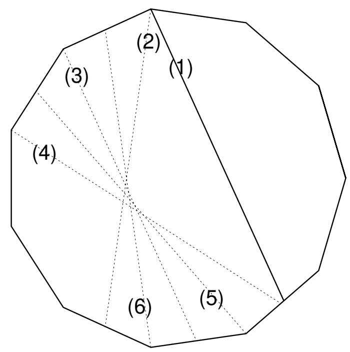

FIGURE 7. Illustration of the perpendiculars (dashed lines) that do not cross one marked perpendicular (solid perpendicular) inside the n-gon, where n is odd.

perpendicular to edge j ends at vertex $j+1+n/2$; the perpendicular starting at vertex $\vec{v}_j$ perpendicular to edge $j+1$ ends at vertex $j-1+n/2$. *Inside* the n-gon, each of the n perpendiculars is intersected by $n-4$ perpendiculars. The 4 perpendiculars out of n that do *not* hit a perpendicular are (see Figure 8):

(1) The perpendicular itself
(2) The perpendicular that runs parallel to it, starting at vertex $j+1$ or $j-1$.
(3) The perpendicular that starts at the same vertex and perpendicular to the other edge.
(4) The perpendicular that starts where this perpendicular ends.

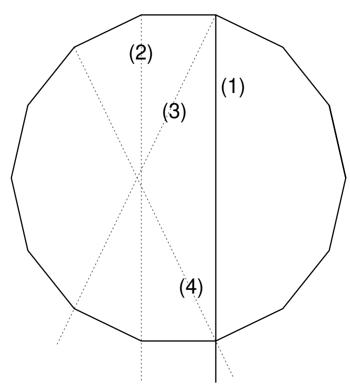

FIGURE 8. Illustration of the perpendiculars (dashed lines) that do not cross one marked perpendicular (solid perpendicular) inside the n-gon, where n is even.

That splits each of the perpendiculars into $n-3$ pieces inside the n-gon, a total of $n(n-3)$ edges of polygons inside the n-gon. Adding those n at the perimeter we count $E = n+n(n-3) = n^2-2n$ edges. The $n-4$ crossings counted above on each diagonal define $n(n-4)/2$ vertices inside the n-gon; we divide by 2 here because there are n diagonals but each crossing would be counted separately (twice) for each perpendicular. Adding the n vertices along the perimeter considered above,

$V = n + n(n-4)/2 = \frac{1}{2}n^2 - n$. Formula (4) yields $F = 1 + (n^2 - 2n) - (\frac{1}{2}n^2 - n) = \frac{1}{2}n^2 - n + 1$, in accordance with (3).

5. Summary

Inspired by the problem of finding places of a hut on a n-regular island we have dissected the regular n-gon into tiles defined by perpendiculars to sides, and found (3) as the number of tiles.

Appendix A. Coordinate Arithmetics

We enumerate the vertices of the n-gon from 0 to $n-1$ and put them at the n-th root of unity in the complex number plane, i.e., at the Cartesian coordinates

$$\vec{v_j} = \begin{pmatrix} \cos(j\omega) \\ \sin(j\omega) \end{pmatrix}, \quad j = 0 \dots n-1, \tag{5}$$

where

$$\omega \equiv 2\pi/n \tag{6}$$

is the angle covered by each polygon side observed from the center. The formula expresses that the line from the polygon center to vertex j has the inclination angle $j\omega$, if inclinations are measured counterclockwise towards the horizontal, and that the outcircle radius of the polygon has been fixed to 1. The triangle formed by the center of the polygon and two adjacent vertices has interior angles of ω at the center and $(\pi - \omega)/2$ at the edge of the polygon: Figure 9.

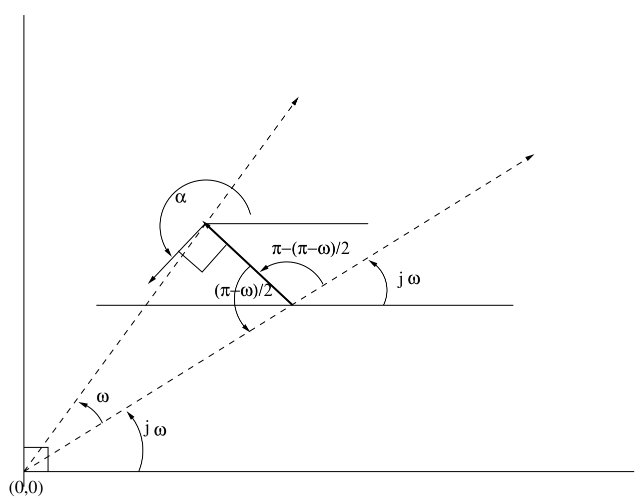

Figure 9. $j\omega$ is the direction from the center to the vertex j. Further rotation by ω yields the direction to vertex $j+1$. The angle ω and the two angles $(\pi - \omega)/2$ inside the wedge defined by edge j add to 180°. The perpendicular to edge j has the inclination α_j.

We also enumerate the edges of the n-gon: edge number j is the edge starting at vertex j and ending at vertex $j+1$ (modulo n). The angle of the inclination of edge j equals the 180°-complement of the inner wedge angle plus the inclination of the vertex: $\pi - (\pi - \omega)/2 + j\omega$. The perpendicular has an inclination that is $\pi/2$ larger than this. In summary, the perpendicular to edge j at vertex j or at vertex $j+1$ has the inclination

$$\alpha_j \equiv \pi/2 + \pi - (\pi - \omega)/2 + j\omega = \pi + \left(\frac{1}{2} + j\right)\omega. \tag{7}$$

Appendix B. Intersections with Opposite Side

The sketch in Figure 10 shows where the perpendiculars hit the opposite edges if n is odd. (For n odd, the flips of the D_{2n} group of the symmetry of the polygon are edge-to-vertex, for n even edge-to-edge.) The wedge angle ω of Figure 9 becomes $\omega/2$ for a half-edge.

Remark 3. *Figure 10 shows that the incircle radius of the n-gons is* $\cos(\omega/2)$ *and the that the incircle radius of the polygons described in* (1) *is* $\sin(\omega/2)$.

Projection to the opposite sides defines a right triangle with a leg of length $\sin(\omega/2)$ and an inner angle of $\omega/2$, such that the hypothenuse is $h = \tan(\omega/2)$ and the other cathetus $h' = \tan(\omega/2)\sin(\omega/2)$. Because $\tan(\omega/2) > \sin(\omega/2)$, more than half of the edge length, two perpendiculars do *not* meet at opposite edges; eventually the number of vertices and edges relevant to Euler's formula is indeed $3n$ for odd n, as claimed in the main text.

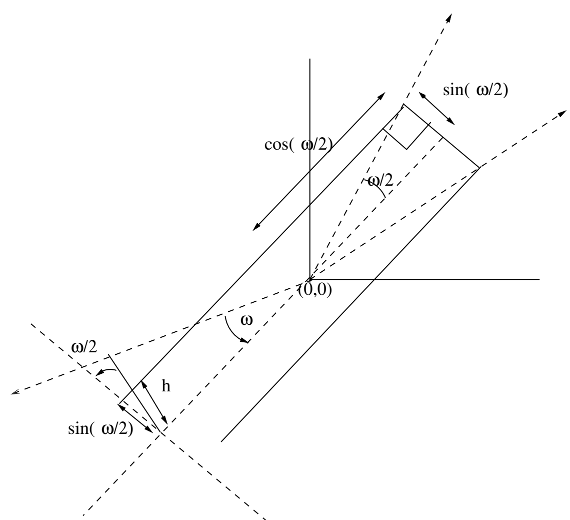

Figure 10. The half side length of the n-gon is $\sin(\omega/2)$. For odd n, we project the two perpendiculars and a half edge to the opposite side of the polygon where they hit two opposite edges. The perpendiculars chop off pieces of length h of these opposite sides.

Appendix C. Multi-crossings

While counting crossings in the previous sections we have tacitly assumed that each pair of crossings of perpendiculars creates a single crossing. (For general diagonals in the n-gon the calculation is more complicated [7, 8].) To complete the proof we need to ensure that there are no points where 3 or more perpendiculars meet inside the n-gon.

Because the direction of the perpendicular is $p_j = \begin{pmatrix} \cos\alpha_j \\ \sin\alpha_j \end{pmatrix}$, a point on the perpendicular starting at vertex j perpendicular to edge j has the form $\vec{v}_j + t_j\vec{p}_j$, where $t_j \geq 0$ is the parameter. A point on the perpendicular starting at vertex $j+1$ perpendicular to edge j has the form $\vec{v}_{1+j} + t_j\vec{p}_j$.

Remark 4. *Figure 10 shows that the range for the parameter t, the value of the parameter before the perpendicular exits the n-gon at the opposite side, is $t \leq 2\cos(\omega/2)$ for even n and $t \leq 1 + \cos(\omega/2) - h' = 1 + \cos(\omega)/\cos(\omega/2)$ for odd n.*

A point where two perpendiculars starting at side j and side k meet has the common representation

$$\vec{v}_{j+\delta_j} + t_j\vec{p}_j = \vec{v}_{k+\delta_k} + t_k\vec{p}_k \tag{8}$$

where the two $\delta_{j,k}$ are either 0 or 1 (Figure 11).

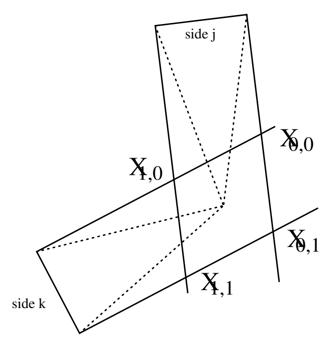

FIGURE 11. Two lanes of the perpendiculars defined by two edges j and k intersect at a parallelogram inside the n-gon. The four combinations of δ_j and δ_k select one of the four intersections X_{δ_j,δ_k} of the two perpendiculars (which of the four corners of the parallelogram).

This is a linear system of 2 equations with two unknowns t_j and t_k if written down for the two Cartesian coordinates, which is solved either by Cramer's rule or by building the cross product with $\vec{p}_k$ on both sides and considering the z-component:

$$t_j = \frac{[(\vec{v}_{k+\delta_k} - \vec{v}_{j+\delta_j}) \times \vec{p}_k]_z}{[\vec{p}_j \times \vec{p}_k]_z} \tag{9}$$
$$= \frac{[\cos((k+\delta_k)\omega) - \cos((j+\delta_j)\omega)]\sin\alpha_k - [\sin((k+\delta_k)\omega) - \sin((j+\delta_j)\omega))]\cos\alpha_k}{\cos\alpha_j \sin\alpha_k - \sin\alpha_j\cos\alpha_k}.$$

Standard transformations for sums and products of the trigonometric functions yield

$$
\begin{aligned}
t_j &= \frac{\sin[\alpha_k - (k+\delta_k)\omega] - \sin[\alpha_k - (j+\delta_j)\omega]}{\sin(\alpha_k - \alpha_j)} \\
&= \frac{\sin[(\delta_k - 1/2)\omega] + \sin[(k-j-\delta_j+1/2)\omega]}{\sin[(k-j)\omega]} \\
&= \frac{\cos[(k-j-\delta_k-\delta_j+1)\omega/2]\sin[(k-j+\delta_k-\delta_j)\omega/2]}{\sin[(k-j)\omega/2]\cos[(k-j)\omega/2]} \\
&= \begin{cases} \frac{\cos[(k-j+1)\omega/2]}{\cos[(k-j)\omega/2]}, & \delta_j = \delta_k = 0; \\ \frac{\sin[(k-j+1)\omega/2]}{\sin[(k-j)\omega/2]}, & \delta_j = 0 \wedge \delta_k = 1; \\ \frac{\sin[(k-j-1)\omega/2]}{\sin[(k-j)\omega/2]}, & \delta_j = 1 \wedge \delta_k = 0; \\ \frac{\cos[(k-j-1)\omega/2]}{\cos[(k-j)\omega/2]}, & \delta_j = \delta_k = 1. \end{cases} \\
&= \begin{cases} \frac{\cos[(k-j)\omega/2]\cos(\omega/2) - \sin[(k-j)\omega/2]\sin(\omega/2)}{\cos[(k-j)\omega/2]}, & \delta_j = \delta_k = 0; \\ \frac{\sin[(k-j)\omega/2]\cos(\omega/2) + \cos[(k-j)\omega/2]\sin(\omega/2)}{\sin[(k-j)\omega/2]}, & \delta_j = 0 \wedge \delta_k = 1; \\ \frac{\sin[(k-j)\omega/2]\cos(\omega/2) - \cos[(k-j)\omega/2]\sin(\omega/2)}{\sin[(k-j)\omega/2]}, & \delta_j = 1 \wedge \delta_k = 0; \\ \frac{\cos[(k-j)\omega/2]\cos(\omega)/2) + \sin[(k-j)\omega/2]\sin(\omega/2)}{\cos[(k-j)\omega/2]}, & \delta_j = \delta_k = 1. \end{cases} \\
&= \cos(\omega/2) + \sin(\omega/2) \times \begin{cases} -\tan[(k-j)\omega/2], & \delta_j = \delta_k = 0; \\ \cot[(k-j)\omega/2], & \delta_j = 0 \wedge \delta_k = 1; \\ -\cot[(k-j)\omega/2], & \delta_j = 1 \wedge \delta_k = 0; \\ \tan[(k-j)\omega/2], & \delta_j = \delta_k = 1. \end{cases}
\end{aligned} \tag{10}
$$

The task is to show that for some fixed j and δ_j this t_j is distinct for all indices k, such that the crossings are all well-spread over the perpendicular.

Because the tan and cot are monotonous functions in each branch, because $j-k$ is in the range $1 \ldots n-1$, and because $\omega/2 = \pi/n$, so the values of $(j-k)\omega/2$ are spread over an interval not larger than π, the individual values for each of the 4 branches are all distinct. To check that there are no crossings with two different δ_k, we also must ensure

$$\tan[(k-j)\omega/2] \neq -\cot[(k'-j)\omega/2]; \tag{11}$$

$$\sin[(k-j)\omega/2]\sin[(k'-j)\omega/2] \neq -\cos[(k'-j)\omega/2]\cos[(k-j)\omega/2]; \tag{12}$$

$$\cos[(k'-k)\omega/2] - \cos[(k+k'-2j)\omega/2] \neq -\cos[(k-k')\omega/2] - \cos[(k+k'-2j)\omega/2]; \tag{13}$$

$$\cos[(k'-k)\omega/2] \neq -\cos[(k-k')\omega/2]. \tag{14}$$

This requires that the cosine is not zero. To avoid the zeros, with (6),

$$(k'-k)\omega/2 \neq \pi/2 \wedge (k-k')\omega/2 \neq 3\pi/2; \tag{15}$$

$$k'-k \neq n/2 \wedge k-k' \neq 3n/2. \tag{16}$$

For odd n this is correct because $k - k'$ are integer and $n/2$ and $3n/2$ are not, and for even n these equations are only valid for the degenerate pairs of perpendiculars that share the start and end points and these have already been discarded in Section 4.

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