A317203

F. M. Dekking, Delft University of Technology, F.M.Dekking@math.tudelft.nl

December 28, 2019

Proof of a close relationship between A317203 and A108103

Let $a :=$ A317203 $= 3, 1, 3, 2, 3, 1, 3, 1, 3, 2, 3, 1, 3, 2, 3, 1, 3, \ldots$.

Let A108103 $= 1, 2, 1, 3, 1, 2, 1, 3, 1, 3, 1, 2, 1, 3, 1, 2, 1, 3, 1, 3, 1, 2, \ldots$.
We have: A108103 = Fixed point of the square of the morphism: $1 \to 3, 2 \to 1, 3 \to 121$, starting with 1.
The square of this morphism is $1 \to 121, 2 \to 3, 3 \to 313$.

Let σ be the defining morphism of A108103, but on the alphabet $\{3, 2, 1\}$ instead of $\{1, 2, 3\}$, i.e., we consider the defining morphism composed with the permutation Π given by $\Pi(1) = 3, \Pi(2) = 2, \Pi(3) = 1$.
Then σ is given by

$$\sigma(1) = 131,\ \sigma(2) = 1,\ \sigma(3) = 323.$$

Let $z = 32313231313231323131323\ldots$ be the fixed point of σ starting with the letter 3. We then have

$$z = \Pi(\text{A108103}).$$

Let τ be the morphism associated to $(a(n))$, so τ is given by

$$\tau(1) = 132,\ \tau(2) = 1,\ \tau(3) = 3.$$

Note that $(a(n))$ is not generated by iteration of τ. However, if $y = 1323131323132313132\ldots$ is the infinite fixed point of τ starting with the letter 1, then it is easy to see that

$$a(n + 1) = y(n) \quad \text{for all } n = 1, 2, ...$$

We claim that sequences A317203 and A108103 are closely related:

$$\text{A317203}(n + 1) = y(n) = z(n + 3) = \Pi(\text{A108103}(n + 3)) \quad \text{for all } n = 1, 2, \ldots. \qquad (*)$$

Equation $(*)$ is a consequence of the following THEOREM.

THEOREM $323y = z$.

The theorem follows directly by letting $n \to \infty$ in the following proposition.

PROPOSITION For $n = 0, 1, 2\ldots$

$$323\tau^{2n+1}(1) = \sigma^{n+1}(32)32.$$

For the proof of the PROPOSITION we need LEMMA 1. We extend the concept of a word from the semigroup to the free group, so for example $3(13)^{-1} = 1^{-1}$.

LEMMA 1 For $n = 0, 1, 2\ldots \quad \sigma^n(32) = 321^{-1}\sigma^n(1)$.

Proof: Induction. True for $n = 0$. Suppose true for n. Then

$$\sigma^{n+1}(32) = \sigma^n(3231) = \sigma^n(32)\sigma^n(31) = 321^{-1}\sigma^n(1)\sigma^n(31) = 321^{-1}\sigma^n(131) = 321^{-1}\sigma^{n+1}(1). \quad \square$$

LEMMA 2 For $n = 0, 1, 2 \ldots$

$$13\tau^{2n+1}(1) = \sigma^{n+1}(1)32, \text{ and } \quad 323\tau^{2n+1}(2) = \sigma^{n+1}(3)1.$$

Proof: Simultaneous induction. For $n = 0$ we have $13\tau(1) = 13132 = \sigma(1)32$, and $323\tau(2) = 3231 = \sigma(3)1$. Suppose true for n. Then

$$\begin{aligned}
13\tau^{2n+3}(1) &= 13\tau^{2n+1}(\tau^2(1)) \\
&= 13\tau^{2n+1}(13231) \\
&= 13\tau^{2n+1}(1)\tau^{2n+1}(3)\tau^{2n+1}(2)\tau^{2n+1}(3)\tau^{2n+1}(1) \\
&= \sigma^{n+1}(1)32\,3\,\tau^{2n+1}(2)3(13)^{-1}\sigma^{n+1}(1)32 \\
&= \sigma^{n+1}(1)\sigma^{n+1}(3)1\,3(13)^{-1}\sigma^{n+1}(1)32 \\
&= \sigma^{n+1}(131)32 \\
&= \sigma^{n+2}(1)32.
\end{aligned}$$

Also, applying LEMMA 1 in the fifth step,

$$\begin{aligned}
323\tau^{2n+3}(2) &= 323\tau^{2n+1}(\tau^2(2)) \\
&= 323\tau^{2n+1}(132) \\
&= 323\tau^{2n+1}(1)\tau^{2n+1}(3)\tau^{2n+1}(2) \\
&= 323(13)^{-1}\sigma^{n+1}(1)32\,3\,\tau^{2n+1}(2) \\
&= 321^{-1}\sigma^{n+1}(1)\sigma^{n+1}(3)1 \\
&= \sigma^{n+1}(32)\sigma^{n+1}(3)1 \\
&= \sigma^{n+1}(323)1 \\
&= \sigma^{n+2}(3)1.
\end{aligned}$$

This ends the proof of LEMMA 2. □

Proof of the PROPOSITION:
From LEMMA 1 and LEMMA 2 we have for $n = 0, 1, 2 \ldots$

$$\sigma^{n+1}(32)32 = 321^{-1}\sigma^{n+1}(1)32 = 321^{-1}13\tau^{2n+1}(1) = 323\tau^{2n+1}(1). \quad \square$$

REMARK Since A317203(1) $= 3 = \Pi($A108103(3)$)$, Equation (*) also holds for $n = 0$.