SEQUENCE A309878

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Let $i = \sqrt{-1}$ and consider the sequence $\{b_n\}_{n\geq 0}$ defined recursively as follows: $b_0 = 0$ and, for $n \geq 1$,

$$b_n = (n + b_{n-1})(1+i). \tag{1}$$

Let $B(x)$ denote the generating function of $\{b_n\}_{n\geq 0}$. Let $\{a_n\}_{n\geq 0}$ be the sequence corresponding to the real part of $\{b_n\}_{n\geq 0}$, i.e., for $n \geq 0$ we set $a_n = \mathrm{Re}(b_n)$. Sequence A309878 in the OEIS [1] is defined to be $\{a_n\}_{n\geq 0}$. The purpose of this note is to prove that all the conjectures stated in A309878 hold true. These follow easily from the following explicit formula for a_n.

Theorem 1. *We have*

$$a_n = 2^{\frac{n+2}{2}} \sin\left(\frac{n\pi}{4}\right) - n.$$

Proof. Multiplying (1) by x^n and summing over $n \geq 1$ we obtain

$$\sum_{n\geq 1} b_n x^n = \sum_{n\geq 1} (n + b_{n-1})(1+i)x^n. \tag{2}$$

Since $b_0 = 0$, we obtain

$$\sum_{n\geq 0} b_n x^n = (1+i)x \sum_{n\geq 1} n x^{n-1} + (1+i)x \sum_{n\geq 0} b_n x^n. \tag{3}$$

It follows that

$$B(x) = \frac{(1+i)x}{(1-x)^2} + (1+i)xB(x),$$

from which we conclude that

$$\begin{aligned} B(x) &= \frac{(1+i)x}{(1-x)^2(1-(1+i)x)} \\ &= \frac{-1+i}{(1-x)^2} - \frac{2+2i}{1-(2x+i)} + \frac{1+i}{1-x} \\ &= \frac{-1+i}{(1-x)^2} - \frac{2+2i}{1-i}\frac{1}{1-\frac{2x}{1-i}} + \frac{1+i}{1-x} \end{aligned}$$

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\begin{aligned}
&= (-1+i)\sum_{n\geq 0} nx^n + 2i\sum_{n\geq 0} x^n - \frac{2+2i}{1-i}\sum_{n\geq 0}\frac{2^n}{(1-i)^n}x^n \\
&= \sum_{n\geq 0}\left((n+2)i - n - \frac{2^{n+2}}{(1-i)^{n+2}}\right)x^n \\
&= \sum_{n\geq 0}\left[2^{\frac{n+2}{2}}\sin\left(\frac{n\pi}{4}\right) - n + \left((n+2) - 2^{\frac{n+2}{2}}\sin\left(\frac{(n+2)\pi}{4}\right)\right)i\right]x^n.
\end{aligned}
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It follows that $a_n = 2^{\frac{n+2}{2}}\sin\left(\frac{n\pi}{4}\right) - n$. □

References

[1] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, OEIS Foundation Inc., `https://oeis.org`.