A note on A308739 and A308740

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We show how to find the simple continued fraction expansions of the constants $3c$ and $c/3$ where c is either the constant A308739 or the constant A308740.

1) Suppose the real number X, $0 < X < 1$, has the simple continued fraction expansion

$$X = \frac{1}{x_1 +}\,\frac{1}{x_2 +}\,\frac{1}{x_3 + \cdots}.$$

Further suppose each partial quotient x_i is congruent to 1 (mod 3) and the partial quotients $x_2, x_6, x_{10}, \ldots$ are all greater than 1. Then we can apply the following identity

$$3\left(\frac{1}{x_1 +}\,\frac{1}{x_2 +}\,\frac{1}{x_3 +}\,\frac{1}{x_4}\right)$$

$$= \frac{1}{\frac{x_1-1}{3} +}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{\frac{x_2-4}{3} +}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{\frac{x_3-1}{3} +}\,\frac{1}{3x_4} \tag{1}$$

to immediately write down the simple continued fraction expansion of the number $3X$. (We may also need to use the obvious identity $[0; a_1, \ldots, a_{k-1}, a_k, 0, a_{k+1}, a_{k+2}, \ldots] = [0; a_1, \ldots, a_{k-1}, a_k + a_{k+1}, a_{k+2}, \ldots]$ to remove any partial quotients that are equal to zero.)

The simple continued fraction expansion of the number $\frac{1}{3}X$ can be found by using the identity

$$\frac{1}{3}\left(\frac{1}{x_1 +}\,\frac{1}{x_2 +}\,\frac{1}{x_3 + \cdots}\right) = \frac{1}{3x_1 + 3\left(\frac{1}{x_2 +}\,\frac{1}{x_3 + \cdots}\right)} \tag{2}$$

in conjunction with (1).

For example, the constant $c :=$ A308739 has the simple continued fraction expansion $[0; 1, 4, 7, 10, \ldots, 3k+1, \ldots]$. Repeated application of (1) yields the continued fraction expansion

$$3c = [2; 2, 2, 2, (36k+30, 4k+4, 2, 1, 4k+4, 1, 2, 4k+6)_{k \ge 0}].$$

Applying (2) to the continued fraction expansion of c, and then applying (1) repeatedly to the result, yields the continued fraction expansion

$$\frac{1}{3}c = [0; (36k+3, 4k+1, 2, 1, 4k+1, 1, 2, 4k+3)_{k \ge 0}].$$

2) As before, let X be a real number, $0 < X < 1$, with the simple continued fraction expansion

$$X = \frac{1}{x_1 +}\,\frac{1}{x_2 +}\,\frac{1}{x_3 \ +\cdots}\ .$$

Suppose now each partial quotient x_i is congruent to $2 \pmod 3$. Then we can use the following identity

$$3\left(\frac{1}{x_1 +}\,\frac{1}{x_2 +}\,\frac{1}{x_3 +}\,\frac{1}{x_4}\right)$$
$$= \frac{1}{\frac{x_1-2}{3} +}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{\frac{x_2-2}{3} +}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{\frac{x_3-2}{3} +}\,\frac{1}{3x_4} \qquad (3)$$

to immediately write down the simple continued fraction expansion of the number $3X$.

For example, the constant $c :=$ A308740 has the simple continued fraction expansion $[0; 2, 5, 8, 11, ..., 3k + 2, ...]$. Repeatedly applying (3) to this expansion yields the continued fraction expansion

$$3c = [1; 2, 1, 2, 1, 2, (36k + 33, 4k + 4, 1, 2, 4k + 5, 2, 1, 4k + 6)_{k\geq 0}].$$

Applying (2) and (3) yields the continued fraction expansion

$$\frac{1}{3}c = [0; (36k + 6, 4k + 1, 1, 2, 4k + 2, 2, 1, 4k + 3)_{k\geq 0}].$$