Clustering of primes relative to factors of composites
OR
A variant of Ulam's spiral
(A work in progress)

BY

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Notation:

- $\{p_n\}_{n=1}^{\infty}$ is the sequence of primes:
  $p_1 = 2, p_2 = 3, p_3 = 5, \ldots$.
- $S_n = (p_n, p_{n+1}) \cap \mathbb{N}$ is the set of positive integers x, necessarily composite, with $p_n < x < p_{n+1}$:
  $S_1 = \emptyset$, $S_2 = \{4\}$, $S_3 = \{6\}$, $S_4 = \{8, 9, 10\}, \ldots$.
- $a(n) =$ the total number of prime factors of all elements of S_n (counting multiplicities): $a(1) = 0,\ a(2) = 2,\ a(3) = 2,\ a(4) = 7, \ldots$.
- A table for $a(n)$ for n up to 10,000 can be found at

  `https://oeis.org/search?q=0,2,2,7,3,8&sort=&language=&go=Search`

Algorithm:

- Place a dot, representing p_1, on the unit circle at (1,0), or equivalently, at angle 0 degrees, and label this point as the complex number $\lambda_1 = 1 = e^{i0}$.
- Place a dot, representing p_2, on the unit circle at (-1,0), or equivalently, at angle 180 degrees, and label this point as the complex number $\lambda_2 = -1 = e^{i\pi}$.
- For $n \geq 2$, supposing that dots representing $p_1, \ldots, p_n$ have already been placed on the unit circle, place a dot, representing p_{n+1}, on the unit circle as follows: Starting at the dot representing p_n, proceed counter clockwise around the circle until you reach a dot representing one of $p_1, p_2, \ldots, p_{n-1}$. Next, starting at this latter dot, proceed counter clockwise until you reach another dot representing one of $p_1, p_2, \ldots, p_n$. Continue this process until you have traversed $a(n)$ successive arcs connecting dots. Finally, proceed in a counter clockwise direction one more time to the next dot and place the dot which will represent p_{n+1} at the midpoint of arc just traversed. Label this point as the complex number $\lambda_{n+1} = e^{i\theta_{n+1}}$
- Following this procedure, we would place a dot representing p_3 on the unit circle at (0,-1), or equivalently, at angle

270 degrees, and we would place a dot representing p_4 on the unit circle at the midpoint of the arc connecting the point (-1,0) to the point (0,-1), that is, at $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$, or equivalently, at angle 225 degrees. Thus $\lambda_3 = -i = e^{3\pi/2}$ and $\lambda_4 = e^{5\pi/4}$.

Observation: Let u_n be the number of dots lying on the closed arc from 0 degrees to 45 degrees using the first n primes, and let v_n be the number of dots lying on the closed arc from 180 degrees to 225 degrees using the first n primes. Denote these two arcs by U and V respectively. For example, for the first 16 terms, we have

u_n is the sequence 1,1,1,1,1,1,1,2,3,4,4,4,5,5,6,7, ...
v_n is the sequence 0,1,1,2,3,3,4,4,4,5,5,6,6,7,7,7, ...

The sequences have been found from $n = 1$ to $n = 256$, and is partially summarized in the following table.

n	u_n	v_n	$u_n + v_n$	$(u_n + v_n)/n$	u_n/v_n
8	2	4	6	.7500	0.500
16	7	7	14	.8750	1.000
32	16	14	30	.9375	1.143
64	39	20	59	.9219	1.950
128	78	40	118	.9219	1.950
256	164	72	236	.9219	2.277

Questions:

(1) Does $\lim_{n\to\infty} u_n/v_n$ exist? If so, is it equal to 2?
(2) Does $\lim_{n\to\infty}(u_n + v_n)/n$ exist? If so, is it equal to 1?
(3) Is $\{\lambda_k : k = 1, 2, \ldots\} \cap U$ dense in U, the closed arc from 0 degrees to 45 degrees?
(4) Is $\{\lambda_k : k = 1, 2, \ldots\} \cap V$ dense in V, the closed arc from 180 degrees to 225 degrees?
(5) What is the closure of $\{\lambda_k : k = 1, 2, \ldots\}$ in the unit circle? In particular, are there any limit points outside of $U \cup V$?