Appendix A305402: A note on optical diffraction    Johannes W. Meijer, 26-01-79.

The two-dimensional form of Kirchhoff's formula (Papoulis, p. 302)

1. $$V(x_0, z_0) = \frac{1}{4}\int_{-\infty}^{\infty}\left[u(x,0)\frac{\partial H_0(k\rho)}{\partial n} - H_0(k\rho)\frac{\partial u(x,0)}{\partial n}\right]dx$$

with

2. $$\rho = \sqrt{(x-x_0)^2 + (z-z_0)^2}$$

and Ho(kρ) the Hankel function of the first kind order zero.

According to Papoulis for kρ >> 1

3. $$H_0(k\rho) \sim \sqrt{\frac{2j}{\pi k\rho}}\, e^{jk\rho}$$

with j the imaginary unit.

Furthermore, k = 2π/λ with k the wavenumber and λ the wavelength.

4. $$V(x_0, z_0) = \sqrt{\frac{j}{8\pi k}}\int_{-\infty}^{\infty}\left\{u(x,0)\frac{\partial}{\partial n}\left(\frac{e^{jk\rho}}{\sqrt{\rho}}\right) - \frac{e^{jk\rho}}{\sqrt{\rho}}\frac{\partial u(x,0)}{\partial n}\right.dx$$

We assume that

5. $u(x,z) = u(x)\,e^{jkz}$ and $\frac{\partial}{\partial n} = \frac{\partial}{\partial z}$.

In view of $\frac{\partial F(\rho(z))}{\partial z} = \frac{dF}{d\rho}\frac{d\rho}{dz}$ we find

6. $$\frac{\partial}{\partial z}\left(\frac{e^{jk\rho}}{\sqrt{\rho}}\right) = \frac{e^{jk\rho}}{\sqrt{\rho}}\left(\frac{jk(z-z_0)}{\rho} - \frac{z-z_0}{2\rho^2}\right) \sim \frac{e^{jk\rho}}{\sqrt{\rho}}\,\frac{jk(z-z_0)}{\rho}$$

Because $k(z-z_0)/\rho \gg (z-z_0)/(2\rho^2)$ for kρ >> 1.

7. $$\frac{\partial u(x,0)}{\partial n} = jk\,u(x)$$

8. $$V(x_0, z_0) = \sqrt{\frac{j}{8\pi k}}\int_{-\infty}^{\infty} u(x)\left[-\frac{jkz_0}{\rho} - jk\right]\frac{e^{jk\rho}}{\sqrt{\rho}}\,dx$$

We can write this formula as

9. $$V(x, z_0) = \int_{-\infty}^{\infty} u(x)\, g(\rho(x))\, e^{jk\rho(x)}\,dx$$

with

10. $$g(\rho(x)) = \frac{1}{2\sqrt{j\lambda\rho}}\left(1 + \frac{z_0}{\rho}\right)$$

We assume that

11. $$u(x) = \int_{-\infty}^{\infty} U(f)\, e^{j2\pi fx}\,df$$

12. $$V(x_0, z_0) = \int_{-\infty}^{\infty} U(f)\left[\int_{-\infty}^{\infty} g(\rho(x))\, e^{jk\rho(x) + j2\pi fx}\,dx\right]df$$

We define

13. $$I(f) = \int_{-\infty}^{\infty} g(\rho(x))\, e^{jk\rho(x) + j2\pi fx}\,dx$$

14. $$\varphi(x) = 2\pi fx + k\rho(x)$$

With xsf the point of stationary phase we can write

15. $$\varphi(x) = \varphi(x_{sf}) + (x - x_{sf})\varphi'(x_{sf}) + \frac{(x-x_{sf})^2}{2!}\varphi''(x_{sf}) + \cdots$$

For the point xsf we know that,

16. $$\varphi'(x_{sf}) = 0$$

so we find, with z=0, the following formulae.

17. $$x_{sf} = x_0 - \frac{\lambda f z_0}{\sqrt{1-\lambda^2 f^2}}$$

18. $$\rho(x_{sf}) = \frac{z_0}{\sqrt{1-\lambda^2 f^2}}$$

19. $$\varphi(x_{sf}) = 2\pi f x + k z_0 \sqrt{1-\lambda^2 f^2}$$

20. $$\varphi''(x_{sf}) = \frac{k}{z_0}\left(1-\lambda^2 f^2\right)^{3/2}$$

This leads to the following formula

21. $$I(f) = g\left(\rho(x_{sf})\right) e^{j\varphi(x_{sf})} \int_{-\infty}^{\infty} e^{j\frac{k}{2z_0}\left(1-\lambda^2 f^2\right)^{3/2}\left(x-x_{sf}\right)^2} dx$$

For the Gaussian integral it is known that

22a. $\int_{-\infty}^{\infty} e^{jy^2} dy = \sqrt{\pi}\, e^{j\pi/4}$ and consequently 22b. $\int_{-\infty}^{\infty} e^{jpy^2} dy = \sqrt{\frac{j\pi}{p}}$

which leads to

23. $$I(f) = \frac{1}{2}\left(1 + \frac{1}{\sqrt{1-\lambda^2 f^2}}\right) e^{jkz_0\sqrt{1-\lambda^2 f^2}}\, e^{j2\pi f x}$$

so we conclude that

24. $$V(x_0, z_0) = \int_{-\infty}^{\infty} U(f)\left[\frac{1}{2}\left(1 + \frac{1}{\sqrt{1-\lambda^2 f^2}}\right) e^{jkz_0\sqrt{1-\lambda^2 f^2}}\right] e^{j2\pi f x}\, dx$$

We define the optical transfer function $H_{FK}(f)$

25. $$V(x_0, z_0) = \int_{-\infty}^{\infty} U(f)\, H_{FK}(f)\, e^{j2\pi f x}\, df$$

(F=Fresnel and K=Kirchhoff).

26. $$H_{FK}(f) = \frac{1}{2}\left(1 + \frac{1}{\sqrt{1-\lambda^2 f^2}}\right) e^{jkz_0\sqrt{1-\lambda^2 f^2}}$$

It is well-known that (Goodman, p. 60).

27. $$H_{RS1}(f) = e^{jkz_0\sqrt{1-\lambda^2 f^2}}$$

(R=Rayleigh and S=Sommerfeld).

We conclude that

28. $$H_{RS2}(f) = \frac{1}{\sqrt{1-\lambda^2 f^2}}\, e^{jkz_0\sqrt{1-\lambda^2 f^2}}$$

because the Fresnel-Kirchhoff solution is the average of the two Rayleigh-Sommerfeld solutions (Goodman, p. 50).

References.

J.W. Goodman, Introduction to Fourier Optics, 1996.

A. Papoulis, Systems and Transforms with Applications in Optics, 1968.