An asymptotic lower bound for A305233

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Let us consider natural k in the following problem:
Find minimal k such that

$$\binom{k}{\lfloor k/2 \rfloor} \geq n.$$

We can take the following asymptotic expansion[1] (see also [1]):

$$\binom{k}{\lfloor k/2 \rfloor} = \frac{2^k}{\sqrt{\frac{\pi}{2}k}}(1 + O(1/k)).$$

Let us assume that $n = \frac{2^k}{\sqrt{\frac{\pi}{2}k}}(1 + O(1/k))$ and express k.

$k = \log_2 \sqrt{\frac{\pi}{2}k} + \log_2 n - \log_2(1 + O(1/k)) = \log_2 \sqrt{\frac{\pi}{2}} + \log_2 n + 1/2 \log_2 k + O(1/k)$.

Let us apply the bootstrap technique assuming that $k = \log_2 \sqrt{\frac{\pi}{2}}n + O(\log_2 k)$.

We get

$$k = \log_2 \sqrt{\frac{\pi}{2}}n + 1/2 \log_2(\log_2 \sqrt{\frac{\pi}{2}}n + O(\log_2 k))$$

$$k = \log_2 \sqrt{\frac{\pi}{2}}n + 1/2 \log_2(\log_2 \sqrt{\frac{\pi}{2}}n) + 1/2 \log_2(1 + O(\frac{\log_2 k}{\log_2 \sqrt{\frac{\pi}{2}}n}))$$

$$k = \log_2 \sqrt{\frac{\pi}{2}}n + 1/2 \log_2(\log_2 \sqrt{\frac{\pi}{2}}n) + O(\frac{\log_2 \log_2 \sqrt{\frac{\pi}{2}}n}{\log_2 \sqrt{\frac{\pi}{2}}n})$$

Since we started with the inequality, this asymptotic expresses a lower bound for k.

References

[1] Donald E. Knuth, Ilan Vardi, and Rolf Richberg. 6581. *The American Mathematical Monthly*, 97(7):626–630, 1990.

[1] `https://oeis.org/A001405` for $\binom{n}{\lfloor n/2 \rfloor}$, $a(n) \sim 2^n/\sqrt{\pi n/2}$ by Charles R Greathouse IV, Oct 23 2015