A DFA FOR ENUMERATING EVEN-ORDER IRREDUCIBLE DIAGRAMS MODULO 3

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Let a_n denote the n^{th} term in the OEIS sequence with serial number A000699. For $n \in \mathbb{N}$, a_n is defined as the number of irreducible diagrams with $2n$ nodes. This is a very interesting integer sequence with applications in quantum field theory and Kreimer's Hopf algebra of rooted trees. It seems natural to consider combinatorial and number-theoretic properties related to the Fredholm–Rueppel-like sequence

$$(a_n (\text{mod } 3) : n \in \mathbb{N}) = (1, 1, 1, 0, 2, 1, 0, 0, 1, 0, 0, 0, 0, 1, 2, 0, 0, 1, 0, 0, \cdots)$$

obtained by reducing the OEIS sequence A000699 modulo 3.

Theorem 1. *For $n \in \mathbb{N}$, we have that:*

$$a_n \equiv_3 \begin{cases} 1 & \text{if the ternary expansion of } n \text{ is of the form } 1\underbrace{00\cdots0}_{\geq 0} \\ & \text{or is of the form } \underbrace{11\cdots1}_{\text{even}, \geq 0}\ 2\underbrace{00\cdots0}_{\geq 0}; \\ 2 & \text{if the ternary expansion of } n \text{ is of the form } \underbrace{11\cdots1}_{\text{odd}}2\underbrace{00\cdots0}_{\geq 0}; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. As an inductive hypothesis, suppose that the above theorem holds for all $m \leq n$ for some $n \in \mathbb{N}$. The corresponding base case holds trivially.

Let $i \in \mathbb{N}$ be an index as given in the summation

$$\text{A000699}_n = (n-1) \sum_{i=1}^{n-1} \text{A000699}_i \text{A000699}_{n-i}.$$

Our inductive proof largely amounts to case analysis.

Case 1: Suppose that the ternary expansion of n is of the form $1\underbrace{00\cdots0}_{\ell_1 \geq 0}$.

Suppose that the expression $\text{A000699}_i \text{A000699}_{n-i}$ does not vanish. By our inductive hypothesis, the ternary expansions of A000699_i and A000699_{n-i} each must be of one of the following forms:

(a) $1\underbrace{00\cdots0}_{\geq 0}$;

(b) $\underbrace{11\cdots1}_{\text{even}, \geq 0}\ 2\underbrace{00\cdots0}_{\geq 0}$;

(c) $\underbrace{11\cdots1}_{\text{odd}}2\underbrace{00\cdots0}_{\geq 0}$;

Since the ternary expansion of i plus the ternary expansion of $n-i$ must yield $1\underbrace{00\cdots0}_{\ell_1 \geq 0}$, it is easily seen that either the expansion $n-i$ is of the form $2\underbrace{00\cdots0}_{\ell_2 \geq 0}$ and $\ell_1 = \ell_2 + 1$ and the expansion of i is of the form $1\underbrace{00\cdots0}_{\ell_2 \geq 0}$, or vice-versa.

So, in this case, by our inductive hypothesis, we have that the reduction of A000699_n modulo 3 is equal to

$$(n-1)(\text{mod } 3)\,(1 \cdot 1 + 1 \cdot 1)\,,$$

which reduces to $2\,(1 \cdot 1 + 1 \cdot 1)$ modulo 3 Therefore, we have that

$$\text{A000699}_n \equiv_3 = 1$$

in the case whereby the ternary expansion of n is of the form $1\underbrace{00\cdots0}_{\geq 0}$.

Similar case analyses may be applied in the remaining cases, and we leave it as an exercise to verify that our inductive technique may be applied in the remaining cases. □

It is thus easily seen that the DFA given below produces the OEIS sequence A000699 modulo 3.