A Shapley Value for TU-Games with Multiple Membership and Externalities*

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Abstract

In this paper we introduce a new form, *clique function form* (CQFF), of TU-games, that allows for multiple membership and explicit externalities. The new notion is based on graphical representation of connections between agents in a game. We adapt the well-known efficiency, symmetry and linearity axioms to the new setting, and obtain a unique value for superadditive CQFF games. In the last section we study how different restrictions on participants and coalitions affect the total number of graphs relevant to a CQFF game.

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1 Introduction

In his appeal Maskin (2016) asks why the cooperative game theory is being strongly dominated by the non-cooperative approach in modern Economics. The first reasons suggested by Maskin is the absence of externalities in the classical cooperative approach: transferable utility (TU) *characteristic function form* (CFF) games (von Neumann and Morgenstern, 1944). Another one is that, even in allowing externalities extensions researchers are still assuming the formation of the grand coalition, neglecting the possibility of a multiple coalitions outcome. Moreover, in Maskin (2003) he considers the multiple membership as another important, but poorly studied, feature of cooperative games.

Consider a situation, where a number of producers of similar goods decide on how to form coalitions in order to maximize their final payoffs. For simplicity, suppose that any participant can be in coalition with anyone. In such a framework the presence of externalities is natural, since there is a difference between competing against disagregated producers, or against a joint coalition of them. Moreover, it could be profitable for a producer to participate in more than one coalition for the sake of getting all their benefits. However, an increasing number of memberships could be very costly for this producer, since she will need to split her time and resources between all chosen coalitions.

What are the main questions that should be asked about the proposed setting? How can we define a cooperative approach, that will allow us to deal with such problems? What is the expected payoff of any participant of this game, assuming that the grand coalition forms? Which coalitions will finally form as an outcome of this game? In this paper we will answer to the first two questions, i.e. provide a desired concept for cooperative games (structure of which should simplify the way of answering the third question), and derive, for any given game, a vector of expected payoffs for all participants under the hypothesis that grand coalition forms. But before proceeding to our concept, we should carefully trace the whole evolutionary path of the cooperative approach, since it will have a direct influence on our work.

The first extension of the CFF approach was the concept of *partition function form* (PFF) games developed in Thrall and Lucas (1963). The main reason for this was the mentioned presence of externalities among different coalitions, which allows us to reproduce such economic concepts as a bargaining power of a coalition, conditional on the outside coalitions structure, or a free-rider problem (through a positive externalities for lonely agents). Since then a great deal of work was done, in particular in Hart and Kurz (1983), where the authors proved, using an axiomatic approach, that the *Owen value* (Owen, 1977) is a reasonable extension of the

Shapley value (Shapley, 1953a) (the most famous single-valued solution for TU-CFF games) on the PFF approach. Moreover, they proposed a number of different notions for the stability of formed coalitions, which were based on various external players' strategies, like the core and the domination. Another work on the core is Kóczy (2000), where the core of a PFF game was considered as a generalization of the core of a CFF game with the use of the best and the worst case scenarios of a residual player's behaviour (the concept was defined by induction). One of the latest papers on the core of PFF game is Bloch and van den Nouweland (2014), where an extended axiomatization of the expectation formation rules and according definitions of the core were developed. The authors highlight the one based on the projection rule, mentioned in Hart and Kurz (1983), since the core defined this way satisfies most of the proposed axioms.

Another significant part of studies was dedicated to the extension of the classical Shapley value, like the Owen value mentioned above. One of the first attempts was made by Myerson (1977), where the author formulated three simple axioms and derived a corresponding value.[1] However, in Pham Do and Norde (2002) the authors present different formula for calculating the Shapley value of PFF games, using four basic axioms and a linear decomposition in unanimity games. An alternative approach to this issue with an explicit use of processes and scenarios of the coalition formation was performed in Grabisch and Funaki (2012). However, an assumption of equal probabilities for different coalition formation orders should be relaxed in order to develop a more general approach. That was done in Faigle and Grabisch (2012), where the authors introduced a Markovian value – an extension of the Shapley value through the Markov coalition formation process.

One more variation of TU-games was introduced in Myerson (2003) by means of *communication games*. The main idea was to define a cooperative game on a particular network (games on networks are widely discussed in Jackson (2008)), which will allow only linked players to affect payoffs of each other. A motivation for this concept arises from a simple fact that *"most decisions that people make, from which products to buy to whom to vote for, are influenced by the choices of their friends and acquaintances"* (Jackson and Zenou, 2012). That can be naturally broadened on the cooperative framework through the logic that coalitions can only function to the extent that they can communicate. In his paper Myerson proposes a natural extension of the Shapley value to communication games: the *Myerson value*. This concept allows us to have another way to capture intercoalitional externalities: coalitions' worths will vary with different network structures.

[1] Hereinafter we will refer to this value as *value, proposed by Myerson,* in order to distinguish it from the Myerson value for communication games mentioned below.

The rest of the paper is structured as follows. In Section 2 we summarize the CFF and PFF concepts and introduce a more general one: *Clique Function Form* (CQFF). In Section 3 we derive a value for these more general games using the axiomatic approach. Section 4 studies sizes of domains of these general games under different restrictions. Concluding Section 5 describes some features of the proposed concept and offers avenues for future research. In Appendix A at the end of the paper the reader will find basic and well-knows definitions, axioms and results related to values for CFF and PFF games.

2 Extension

Although the mentioned concepts of CFF and PFF games possess very important features, they still lack some significant points: explicit externalities in games with graphical representation and a possibility of multiple membership (as was proposed in Maskin (2003)). For example, in addition to the mentioned production setting, the same country can participate both in European Union and United Nations, or one university can be involved in two different research associations. Our goal is to develop a more general concept that will capture these properties and could serve as a basis for an endogenous coalition formation. Before proceeding to our proposal let us briefly introduce the CFF and PFF games concepts:

- *General notions*:
  - $N = \{1, \ldots, n\}$ is the set of all players;
  - $C(N) \equiv 2^N$ is the set of all coalitions in N with a typical element S.
- *Characteristic function form game* is a mapping $v(N) : C(N) \to \mathbb{R}$, where $v(\emptyset) = 0$.
- *Partition function form game*:
  - $\Pi(N) \equiv \{(S^1, \ldots, S^k) \,|\, k \in \{1, \ldots, n\}; S^i \in C(N)\backslash\{\emptyset\}, S^i \cap S^j = \emptyset \ \forall i \neq j \in \{1, \ldots, k\}; \cup_i S^i = N\}$ is the set of all partitions of N with a typical element P;
  - $EC(N) \equiv \{(S, P) \,|\, S \in P; P \in \Pi(N)\}$ is the set of all embedded coalitions (pairs of a partition and one of the corresponding coalitions);
  - PFF game is a mapping $w(N) : EC(N) \to \mathbb{R}$.

The title for our concept is *Clique Function Form* (CQFF) game. Basically, the idea is to develop a cooperative game with a network structure, multiple membership feature and explicit externalities, which could be easily applied on an already existing structure of links between players. For instance, consider a problem of allocating workers to various areas of a project, or researchers to different scientific studies. In the context of this example, possible coalitions could be restricted by a predetermined network structure. Which could be justified, for instance, by geographical or qualification issues, such as: some workers could live in different parts of the world or could have not enough skills to work in some areas of a project.

In order to give a formal definition of a CQFF game we need firstly to introduce one notion from graph theory. A *clique of a graph* is any of its full (fully interlinked) sub-graphs, and a *maximal clique of a graph* is a clique that cannot be included in any other clique, i.e. be a sub-graph of any other clique.

- *Clique function form game*:
  - $\widetilde{\Omega}(N) \equiv \{(S^1, \ldots, S^k) \,|\, S^i \in C(N)\backslash\{\emptyset\}, \forall i \in \{1, \ldots, k\} : \cup_i S^i = N; \nexists i \neq j \in \{1, \ldots, k\} : S^i \subseteq S^j\}$ is the set of all *coalition configurations* (see Albizuri (2010));[2]
  - G_D is a graph on n labeled nodes based on a coalition configuration $D \in \widetilde{\Omega}(N)$ such that any two players (nodes) $i, j \in N$ are linked in G_D if and only if there exists a coalition S in D, containing them, $\{i, j\} \subseteq S \in D$;[3]
  - $\Omega(N) \equiv \{D \,|\, D \in \widetilde{\Omega}(N), D$ is the set of all maximal cliques of its corresponding graph $G_D\}$ is the set of all *coalition combinations* with a typical element D;[4]
  - $IC(N) \equiv \{(S, D) \,|\, S \in D; D \in \Omega(N)\}$ is the set of all *integrated coalitions*;
  - CQFF game is a mapping $q(N) : IC(N) \to \mathbb{R}$.

Simply put, the structure of a CQFF game allows us to have a unique representation of any not-directed graph on n nodes through a coalition combination and vice versa, where the presence of (S, D) in the domain of the game tells us that, under this coalition combination, D, all players in S are interlinked, and $q(S, D)$ gives us the worth of the coalition S under the

[2]In Albizuri (2010) and Albizuri et al. (2006) the authors provide extended to the coalition configurations setting versions for the value, proposed by Myerson, and for the Owen value, correspondingly. Our approach differs from theirs in a geometrical sence: we focus more on the graphical representation of coalitions.

[3]Notice that, a given graph could correspond to different coalition configurations. For instance, two coalition configurations $D = \{\{1, 2, 3\}\}$ and $D' = \{\{1, 2\}, \{2, 3\}, \{3, 1\}\}$ will have the same corresponding triangle graph $G_D = G_{D'}$.

[4]Consider again two coalition configurations with the same corresponding graph: $D = \{\{1, 2, 3\}\}$ and $D' = \{\{1, 2\}, \{2, 3\}, \{3, 1\}\}$. The first one, D, is a coalition combination, while the second one, D', is not, since only D is the set of all maximal cliques of a triangle graph.

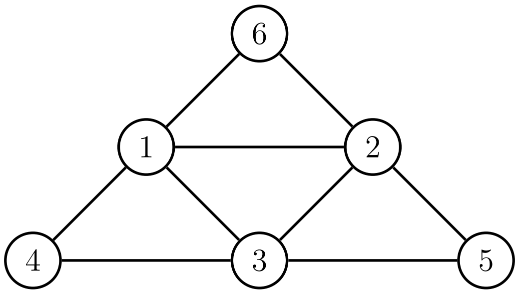


Figure 1

graph G_D (there should be $2^{n(n-1)/2}$ different types of D, as the number of all not directed graphs on n labeled nodes).

In more details, putting into words the definition of the set of all coalition combinations $\Omega(N)$, we want to consider only those coalition configurations, which explicitly describe all existing coalitions. Restricting the initial set of coalition configurations $\widetilde{\Omega}(N)$ in such a way raises a bijection between the restricted set, $\Omega(N)$, and the set of all possible not-directed graphs on n labeled nodes. Through this framework we describe a CQFF game by giving worths to contracts that depict a given graphical structure. Since, if a set of agents S is fully interlinked, it is logical to assume that they all need to sign only one contract in order to determine all relationships between them.

As an illustration of our logic consider the graph on Figure 1. As we can see this graph could be fully explained through only three fully interlinked sub-graphs: $\{1,3,4\}$, $\{2,3,5\}$, $\{1,2,6\}$, since they will give us all needed links. However, under such graphical structure players 1, 2 and 3 are also connected with each other, hence they also form a coalition. As a result, the corresponding coalition combination has four coalitions: $\{1,3,4\}$, $\{2,3,5\}$, $\{1,2,6\}$ and $\{1,2,3\}$. This coalition combination is the set of all maximal cliques of this graph.

To make it more explicit, we present another example of such a unique pair of a coalition combination D and a graph G_D: $D = \{\{1,2,4,5\},\{1,2,3,4\},\{3,4,8\},\{4,6,7\},\{9,10\}\}$ and the graph G_D on Figure 2.

If, as it was mentioned above, before the game starts there is already a graphical structure G, consisting of all pairs of nodes that could potentially be interlinked, the set of all coalition combinations $\Omega(N)$ (the domain of a CQFF game) should be restricted to the set, where there does not exist $D \in \Omega(N)$, such that there exists a pair $i,j \in N : \{i,j\} \subseteq S \in D$ for some $S \in D$, and $\{i,j\} \notin G$. Putting into words, in the case of predetermined network structure G, for a CQFF game we will consider only those coalition combinations which do not

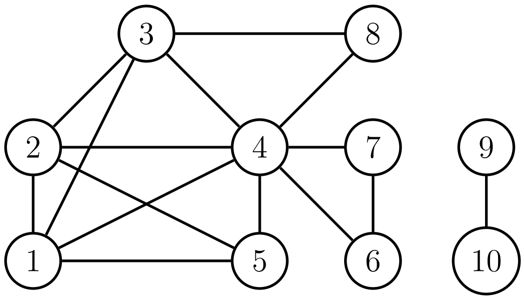


Figure 2

imply that two agents are linked, if they can not be linked under the given graph G.

Summing up the subtotal, the CQFF framework explicitly introduces externalities in cooperative games with multiple membership. Clearly, now any agent can participate in more than one coalition, since a coalition combination is not a partition. Moreover, one coalition may have different worths under one or another coalition combination. So, the way other coalitions have formed may affect the total worth received by the current coalition, which, in turn, rises an opportunity for the presence of externalities.

In addition, the graphical base of the CQFF game concept should simplify the way of constructing and interpreting an endogenous coalition formation process, since any new or deleted link now leads to a particular D with given worths $q(S, D)$ for every $S \in D$. Hence, the procedure of adding/deleting a link could be considered as one step of the endogenous coalition formation algorithm.

3 A Shapley Value for CQFF games

Generally speaking, a value φ of a cooperative game on n players is a vector of size n describing how the worth of the grand coalition should be divided between all participants. Namely, for each CQFF game $q(N) : IC(N) \to \mathbb{R}$ from the domain $dom(\varphi)$ of a value $\varphi : dom(\varphi) \to \mathbb{R}^n$, the following holds $\sum_{i \in N} \varphi_i(q(N)) = q(N, \{N\})$.

In order to talk about an extension of the Shapley value, in general, we should assume that the grand coalition N always forms, for this to happen we need to define the notion of superadditivity for CQFF games.

A CQFF game $q(N)$ is *superadditive* if for any coalition combination $D \in \Omega(N)$ the following holds. Take any two disjoint coalitions $S, S' \in D$, then the worth of the merged coalition should be greater or equal than the sum of their worths, $q(S, D) + q(S', D) \leq q(S \cup$

$S', D')$, where $S \cup S' \in D'$, and graph $G_{D'}$ preserves the structure of G_D outside the coalition $S \cup S'$, i.e. if either i or j are not in $S \cup S'$, then i and j are linked in $G_{D'}$ if and only if i and j are linked in G_D. Simply speaking, after adding to the graph G_D only those links that make all players in $S \cup S'$ fully interconnected (maximal clique), the worth of this joint coalition should be greater or equal than the sum of worth of coalitions S and S' under the initial graph G_D. So, if the CQFF game is superadditive then the grand coalition will form, since it will provide the biggest total worth out of all coalition combinations.

Now we define an *intersection of any two coalition combinations* D and D'. This operation could be considered from two points of view: more natural for our approach geometrical point of view, based on graphs, and more classical point of view of sets, which was used in Myerson (1977).

Definition 1. An *intersection of any two coalition combinations* D and D', $D \wedge D'$, is the set of all maximal cliques of the intersection of their corresponding graphs: $G_{D \wedge D'} \equiv G_D \cap G_{D'}$ (we keep only those links which are present in both graphs).[5]

From the point of view of sets we can redefine the intersection as follows.

Definition 2. An *intersection of any two coalition combinations* D and D', $D \wedge D'$, is the set of different coalitions, such that

$$D \wedge D' = \left\{ S \cap S' \,|\, S \in D, S' \in D', \nexists\, \widetilde{S} \in D \text{ and } \widetilde{S}' \in D' \text{ such that } S \cap S' \subset \widetilde{S} \cap \widetilde{S}' \right\}.$$

Proposition 1. *The two definitions of an intersection of any two coalition combinations are equivalent.*

Proof. Given two coalition combinations D and D' let us call two intersections obtained through both definitions as $(D \wedge D')^{(1)}$ and $(D \wedge D')^{(2)}$.

Suppose that a coalition S^* is contained in $(D \wedge D')^{(1)}$. Hence, all links that are connecting any two players from S^* were present in both graphs G_D and $G_{D'}$. Therefore, there should exist two coalitions: S from D and S' from D', both containing the initial S^*.

Moreover, there could not be a pair of coalitions one from D, another from D', with an intersection strictly containing S^*. Because, otherwise, S^* should not be contained in the set of all maximal cliques of an intersection graph $G_{D \wedge D'}$. Hence, S^* is also contained in $(D \wedge D')^{(2)}$.

[5]Notice that, by construction, the intersection of any two coalition combinations will always be coalition combination: $D, D' \in \Omega(N)$ implies $D \wedge D' \in \Omega(N)$. So, the set of all coalition combinations, $\Omega(N)$, is closed with respect to the operation of intersection.

By inverting the above presented logic we get that, if a coalition S^* is contained in $(D \wedge D')^{(2)}$ then it is also contained in $(D \wedge D')^{(1)}$. $\square$

We need both definitions, since the first one will help us to understand the geometrical logic of the following results, while the second one is easier for calculating the intersection itself, and it is closer to the definition of an intersection of partitions under the PFF approach (see the Appendix Section A.2).

Now, given two integrated coalitions $(S, D), (S', D') \in IC(N)$ we will write

$$(S, D) \gg (S', D') \text{ if } S' \subseteq S \text{ and } D \wedge D' = D'.^6$$

Notice that, $D \wedge D' = D'$ if and only if the graph $G_{D'}$ is a sub-graph of G_D, $G_{D'} = G_{D \wedge D'} \equiv G_D \cap G_{D'}$. Also notice that, the relation $\gg$ on the set of all integrated coalitions, $IC(N)$, is transitive.

As the next step we need to introduce the notion of the *carrier* of a given CQFF game. The reason is to define the set of players $\widetilde{S}$, out of all N, such that any member of $\widetilde{S}$ is very important for the worths of integrated coalitions under this game.

Upon reaching our goal of defining the carrier, we will state the *efficiency axiom* that our value should satisfy. It says that only players from the carrier should share all the worth of the grand coalition. While all other players are irrelevant to the game (dummy-players), so they should receive nothing.

3.1 Carrier of a CQFF game

Now let us introduce a notion of the *carrier of a CQFF game* $q(N)$. Again, the carrier is a subset of agents which are the only ones who crucially matter in terms of getting worth. Suppose now that the carrier of a CQFF game $q(N)$ is $\widetilde{S} \subseteq N$. Consider a coalition combination D. Obviously, if coalition $S \in D$ has no agent from the carrier, then $q(S, D) = 0$. However, if the intersection $S' = S \cap \widetilde{S}$ is non-empty, then only the agents from S' should provide all the worth $q(S, D)$. So, if we consider the grand coalition combination $\{N\}$ and the carrier as a strict subset $\widetilde{S} \subset N$, then after separating the carrier with some (there should exist at least one) coalition combination $\widetilde{D}$, containing $\widetilde{S}$ (and obviously some other coalitions), we should obtain that under the new coalition combination the carrier-coalition receives all the worth of the grand coalition, $q\left(\widetilde{S}, \widetilde{D}\right) = q\left(N, \{N\}\right)$, and all other coalitions receive nothing, that is

[6] To illustrate the notion $\gg$, consider the following example for 3 players: $(S, D) = (\{1, 2, 3\}, \{\{1, 2, 3\}\}) \gg (\{1, 2\}, \{\{1, 2\}, \{2, 3\}\}) = (S', D')$.

$q\left(S, \widetilde{D}\right) = 0$ for all $S \in \widetilde{D}, S \neq \widetilde{S}$. But what if, instead of the grand coalition combination $\{N\}$, we have a more general structure with an arbitrary initial coalition combination D?

For the upcoming results we need to pick up a basis for the space of our CQFF games on N. Firstly, we introduce an *operation of a linear combination of two CQFF games*. The sum of two CQFF games $q(N)$ and $q'(N)$ with weights $a, b \in \mathbb{R}$ is a CQFF game $q^*(N)$, such that $q^*(S, D) = a \cdot q(S, D) + b \cdot q'(S, D)$ for any $(S, D) \in IC(N)$.

Now suppose that in a given game we have only one coalition combination providing all the worth. Such games are called *unanimity games*. How should players from the corresponding coalition split this worth under a coalition combination with m different coalitions which include this one? We assume that the initial coalition members will split its worth equally among all those m coalitions since all the participants of the initial coalition do not care about other (dummy) players. Guided by this logic, we set the following *symmetric unanimity games* to be the basis.

Definition 3. A *symmetric unanimity CQFF game* based on the integrated coalition $(S', D') \in IC(N)$ is

$$q_{(S',D')}(S, D) = \begin{cases} 1/m & \text{if } (S, D) \gg (S', D'), \text{ and exactly } m \text{ coalitions from } D \text{ contain } S' \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2. *The set of all CQFF symmetric unanimity games,*[7] *is a basis for the set of all CQFF games.*

Proof. First of all, notice that, the dimension of the space of all CQFF games is equal to $|IC(N)|$. Any CQFF game can be expressed as a vector of worths, one for each integrated coalition. So, the proposed basis-candidate has exactly the needed number of elements.

Now we need to prove that all unanimity games are linearly independent. Let us prove it by contrary. Suppose, that there are $|IC(N)|$ coefficients, not all nulls, that make the corresponding linear combination of the proposed elements being equal to the zero-game $\overline{0}$ (the vector of zeros of length $|IC(N)|$):

$$q^*(N) \equiv \sum_{(S',D') \in IC(N)} \alpha_{(S',D')} q_{(S',D')} = \overline{0}. \tag{1}$$

Now we can have one of two cases.

[7] Hereinafter for simplicity referred to as *unanimity games*, unless otherwise specified.

1. If there is a not-null coefficient $\alpha_{(S'_1,D'_1)}$, for which there does not exist a not-null coefficient $\alpha_{(S'_2,D'_2)}$, such that $(S'_1, D'_1) \gg (S'_2, D'_2)$ and $(S'_1, D'_1) \neq (S'_2, D'_2)$, then we have the following equation

$$q^*(S'_1, D'_1) = \left(\sum_{(S',D') \in IC(N)} \alpha_{(S',D')} q_{(S',D')} \right) (S'_1, D'_1) = \alpha_{(S'_1,D'_1)} \neq 0,$$

which contradicts (1), since zero-game implies zero worth for any coalition combination.

2. If for any not-null coefficient $\alpha_{(S'_1,D'_1)}$ we can find another not-null coefficient $\alpha_{(S'_2,D'_2)}$, such that $(S'_1, D'_1) \gg (S'_2, D'_2)$ and $(S'_1, D'_1) \neq (S'_2, D'_2)$, then we should be able to obtain an infinite number of different integrated coalitions in a chain $(S'_1, D'_1) \gg (S'_2, D'_2) \gg (S'_3, D'_3) \gg \ldots$, since the relation $\gg$ is transitive. This contradicts the fact that the set of all integrated coalitions, $IC(N)$, is finite for any fixed n.

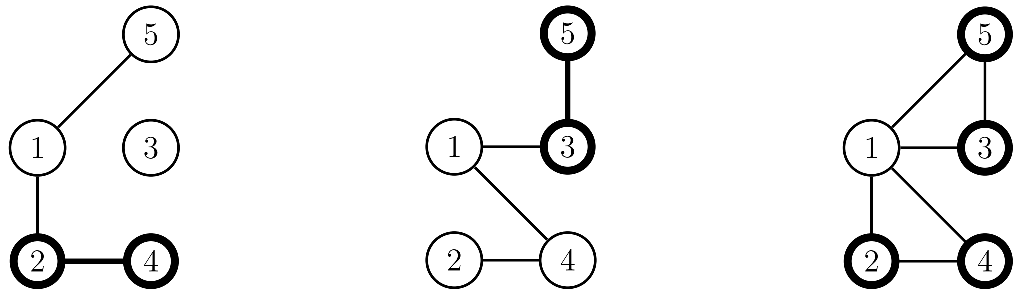
Hence, the proposed set of unanimity games works as a basis for the space of all CQFF games. □

Definition 4. Given a unanimity game $q_{(S,D)}$ we say that S and D are respectively *generative coalition* and *generative coalition combination* of this unanimity game.

Notice that, for any unanimity game its generative coalition should be its carrier. Since only players from the generative coalition provide all the worth in this game.

Now let us define the carrier of a given arbitrary CQFF game, $q(N)$. Take the linear decomposition of $q(N)$ for the above defined unanimity games; that is, $q(N)$ can be uniquely expressed as a linear combination of unanimity games. Suppose that the following unanimity games have not-null coefficients in the decomposition: $q_{(S^1_1,D_1)}, \ldots, q_{(S^1_{l_1},D_1)}, \ldots, q_{(S^k_1,D_k)}, \ldots, q_{(S^k_{l_k},D_k)}$. Consider another coalition combination, D, which graph is a union of graphs corresponding to the generative coalition combinations of these unanimity games $G_D = \cup_{i \in \{1,\ldots,k\}} G_{D_i}$. Finally, *the carrier, $\widetilde{S}$, of the initial game $q(N)$ is the union of all coalitions from D, such that each of them contains at least one generative coalition of the unanimity games in the decomposition: $S^1_1, \ldots, S^1_{l_1}, \ldots, S^k_1, \ldots, S^k_{l_k}$.* Notice that, all agents from all these generative coalitions will be in the carrier, but the carrier, $\widetilde{S}$, may contain more agents.

Definition 5. Given a CQFF game $q(N)$ with the linear decomposition into unanimity games $q_{(S^1_1,D_1)}, \ldots, q_{(S^1_{l_1},D_1)}, \ldots, q_{(S^k_1,D_k)}, \ldots, q_{(S^k_{l_k},D_k)}$, the *carrier of* $q(N)$ is the set of agents



(a) $q_{(\{2,4\},\{\{1,2\},\{1,5\},\{2,4\},\{3\}\})}$ (b) $q_{(\{3,5\},\{\{1,3\},\{1,4\},\{2,4\},\{3,5\}\})}$ (c) $D = \{\{1,2,4\},\{1,3,5\}\}$

Figure 3

$\widetilde{S} \subseteq N$, such that

$$\widetilde{S} = \{\cup S \mid S \in D,\, G_D = \cup_{i\in\{1,\ldots,k\}} G_{D_i},\text{ and } \exists S_j^i,\ \text{s.t. } S_j^i \subseteq S\}.$$[8]

The following example illustrates the logic. Consider a game $q(N)$ with $|N| = 5$, which is a sum of two unanimity games, $q_{(\{2,4\},\{\{1,2\},\{1,5\},\{2,4\},\{3\}\})}$ and $q_{(\{3,5\},\{\{1,3\},\{1,4\},\{2,4\},\{3,5\}\})}$. So, the corresponding to the mentioned above union coalition combination is $D = \{\{1,2,4\},\{1,3,5\}\}$.[9]

According to the definition above, both coalitions from the coalition combination D should be in the carrier, since each of them contains one of the generative coalitions: $\{2,4\} \subset \{1,2,4\}$ and $\{3,5\} \subset \{1,3,5\}$. Hence, all five players are present in the carrier of our game $q(N)$. Why do we need to include also agent 1, while she was not in the generative coalition of any unanimity game? Notice that, for $\{2,3,4,5\}$ to be the carrier we should be able to find some coalition combination, where worths of both unanimity games (total worth of the game) are possessed by coalitions containing only players from the carrier, $\{2,3,4,5\}$. Looking at the coalition combination D we see that it is impossible, since by destroying any link of agent 1 we will *disable* one of the two unanimity games (its generative coalition combination graph will no longer be a subgraph of G_D), and, as a result, lose its worth.

3.2 Game restricted to its carrier

Another very important axiom that the value of a game should satisfy is *symmetry axiom.* An idea is extremely simple, if two agents play exactly the same role in a game, they should receive

[8]Notice that, under the PFF games framework, if we consider only the smallest carrier of a PFF game, the analogous definition (replacing coalition combinations with partitions) of the carrier will be equivalent to one presented in the Appendix Section A.2.

[9]See the reference graphs for the generative coalition combinations of both unanimity games (generative coalitions are in bold) and for D (G_D is a union of generative coalition combinations' graphs) on Figure 3.

exactly the same value regardless of their names or numbers.

Under the previous concepts (CFF and PFF games), where there were no multiple membership, in any unanimity game any player from its generative coalition played exactly the same role in this game. In other words, any unanimity game stayed the same after swapping any two agents from its generative coalition.[10]

Unlike the above, our concept for a CQFF game is more demanding. Consider a unanimity game $q_{(\{1,2\},\{\{1,2\},\{1,3\}\})}$. From above we know that its carrier is its generative coalition, $\widetilde{S} = \{1,2\}$. But if we now swap these agents 1 and 2, we will obtain a new unanimity game $q_{(\{1,2\},\{\{1,2\},\{2,3\}\})}$. So, this swap changed the game.

In order to overcome this obstacle we need to figure out a way to focus our attention only on carrier players of a CQFF game, keeping all their initial influence on the game. Hence, we need to be able to restrict any initial CQFF game to its carrier and obtain a new resulting CQFF game with the following feature. It should include full worth of any unanimity game in the linear decomposition of the initial game. As the first step, we introduce the concept of the *essence of a coalition combination under a given CQFF game*. Then we will formulate *carrier symmetry axiom* by applying the above stated symmetry axiom to the restricted game.

3.2.1 Essence of a coalition combination

Given a CQFF game $q(N)$ and a coalition combination $D \in \Omega(N)$ we construct a coalition combination $\widetilde{D}$, the *essence* of D, as follows. Let $q_{(S_1^1,D_1)},\dots,q_{(S_{l_1}^1,D_1)},\dots,q_{(S_1^k,D_k)},\ \dots,q_{(S_{l_k}^k,D_k)}$ be unanimity games with not-null coefficients in the decomposition of $q(N)$. Now suppose that the following unanimity games' generative coalition combinations' graphs, G_{D_i}, are sub-graphs of our initial graph G_D: $q_{(S_1^1,D_1)},\dots,q_{(S_{l_1}^1,D_1)},\dots,q_{(S_1^m,D_m)},\ \dots,q_{(S_{l_m}^m,D_m)}$, $m \leq k$. Also take a sub-graph of G_D on the nodes $\widetilde{S}$ (carrier of $q(N)$), $G_D|_{\widetilde{S}}$. Finally, the essence of D is $\widetilde{D}$ with the corresponding graph $G_{\widetilde{D}} = (\cup_{i=1,\dots,m} G_{D_i}) \cup \left(G_D|_{\widetilde{S}}\right)$. If the set of sub-graphs is empty ($m = 0$), then we say that the essence of D is $\widetilde{D}$ with the corresponding graph $G_{\widetilde{D}} = \left(G_D|_{\widetilde{S}}\right) \cup G_{\{\{1\},\dots,\{n\}\}}$, so only agents from the carrier could be linked, and they are linked if and only if they are linked in G_D.

So, the essence $\widetilde{D}$ preserves the carrier structure of the initial coalition combination D, and worths of its coalitions are constructed out of worths of the same unanimity games as created all worths in D (since a generative coalition combination graph of a unanimity game is a sub-graph of $G_{\widetilde{D}}$ if and only if it is a sub-graph of G_D). An example of how to construct the

[10]For more extensive description of constructions of values for CFF and PFF games see the Appendix A.

essence is presented in the italics section of step 6 below.

Notice that, under a given CQFF game $q(N)$, two different coalition combinations D and D' may have the same essence $\widetilde{D}$, and each coalition combination D has exactly one essence. Hence, the set of all essences under a given CQFF game $q(N)$, we denote it as $ess(q)$, is the subset of the set of all coalition combinations, $ess(q) \subseteq \Omega(N)$. Also, from the definition of the carrier we can infer that only coalitions from the carrier agents from the essence can posses any not-null worth, while any other coalition from the essence will always have zero worth.

Proposition 3. *Given a CQFF game $q(N)$ with the carrier $\widetilde{S}$, if $\widetilde{D}$ is the essence, $\widetilde{D} \in ess(q)$, then the following two properties hold.*

1. *If a coalition S from $\widetilde{D}$ contains a non-carrier agent (i.e. there exists $i \in S$, such that $i \notin \widetilde{S}$), then its worth should be zero, $q(S, \widetilde{D}) = 0$.*
2. *If $\widetilde{D}$ is the essence of D, then they have the same total worth: $\sum\limits_{S \in \widetilde{D}} q(S, \widetilde{D}) = \sum\limits_{S \in D} q(S, D)$.*

Proof. Let us prove the first statement by contrary. Suppose that there exists such coalition $S \in \widetilde{D} \in ess(q)$ containing a non-carrier agent $a \notin \widetilde{S}$ with a not-null worth, $q(S, \widetilde{D}) \neq 0$. Hence, there exists a unanimity game $q_{(S',D')}$ from the decomposition of $q(N)$, such that its generative coalition combination graph $G_{D'}$ is a sub-graph of $G_{\widetilde{D}}$, and its generative coalition S' is a subset of S. Consider a sub-coalition $\hat{S} \equiv \{a\} \cup S' \subseteq S$ of the initial coalition.

From the definition of the essence we know that a non-carrier structure of $G_{\widetilde{D}}$ consists only of the non-carrier structures of some generative coalition combinations graphs of unanimity games from the decomposition of the initial game: for any two agents $i \in N$ and $j \notin \widetilde{S}$, they could be linked in $G_{\widetilde{D}}$ if and only if they are linked in some generative coalition combination graph of some unanimity game from above. Hence, if we take the union of all generative coalition combinations graphs of unanimity games from the decomposition of the initial game (union graph from the definition of the carrier), we should be able to find a coalition S'' in this graph fully containing our sub-coalition $\hat{S}$. Since $\hat{S}$ contains the generative coalition S' by construction, then coalition S'' also contains the generative coalition S', $S' \subseteq \hat{S} \subseteq S''$. Therefore, from the definition of the carrier we should include all agents from S'' into the carrier. Hence, all agents from our sub-coalition $\hat{S}$ should also be in the carrier. So, player a should be the carrier agent. Contradiction.

The second statement is obvious from the structure of the essence. By construction, a generative coalition combination's graph of any unanimity game from the decomposition of the initial game is a sub-graph of $G_{\widetilde{D}}$ if and and only if it is a sub-graph of G_D. Hence, the whole worth of both D and its essence $\widetilde{D}$ will consist of exactly the same components. $\Box$

3.2.2 Restriction to the carrier

Given a CQFF game $q(N)$ with the carrier $\widetilde{S}$, we can construct a new CQFF game on $\widetilde{S}$ which we will call the *restriction of* $q(N)$ *to its carrier*.

So, a CQFF game $q'(\widetilde{S})$ is a *restriction of* $q(N)$ *to its carrier* $\widetilde{S}$ if the following holds. For any integrated coalition $(S', D') \in IC(\widetilde{S})$ we have $q'(S', D') = q(S', \widetilde{D})$ if the coalition combination $\widetilde{D}$ is the essence of D under $q(N)$, where in G_D a link between agents i and j may be absent only if both i and j are from the carrier $\widetilde{S}$ (all other links are present in G_D), and a graph $G_{D'}$ is the sub-graph of $G_{\widetilde{D}}$ on the nodes $\widetilde{S}$. Proposition 3 guarantees us that the restricted game will capture all the wealths of the original game. Since for any given carrier structure of the coalition combination D we choose this G_D such that, it has the most possible number of generative coalition combinations' graphs of unanimity games from the decomposition of $q(N)$ as sub-graphs.

More explicitly, these are the steps to get the restriction to the carrier of the initial CQFF game $q(N)$ (*each step will be followed by a continuous example in italics*):

1. Decompose $q(N)$ into the linear combination of the unanimity games.

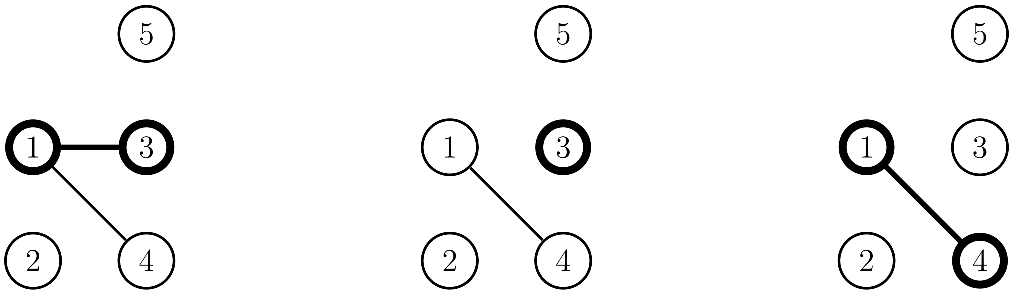
   Suppose that $|N| = 5$ *and our game* $q(N)$ *is a linear combination of the following unanimity games on 5 players,* $q_{(\{1,3\},\{\{2\},\{5\},\{1,3\},\{1,4\}\})}$, $q_{(\{3\},\{\{2\},\{3\},\{5\},\{1,4\}\})}$, $q_{(\{1,4\},\{\{2\},\{3\},\{5\},\{1,4\}\})}$, $q_{(\{1\},\{\{1\},\{2\},\{4\},\{3,5\}\})}$, $q_{(\{1\},\{\{1\},\{2,4\},\{3,5\}\})}$, $q_{(\{3\},\{\{3\},\{1,5\},\{2,4\}\})}$ *with all six coefficients being equal to one.*[11]

2. Take a union of all graphs corresponding to the generative coalition combinations of the obtained above unanimity games.

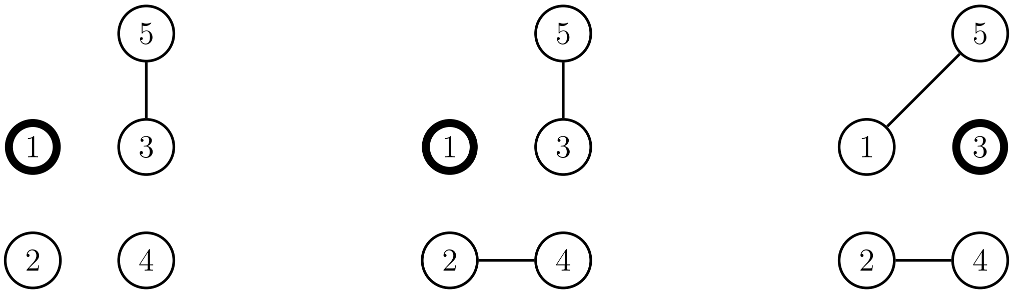
   After the unification of all six graphs corresponding to the generative coalition combinations we obtain a graph G_{D_1} *that is shown on Figure 5.*

3. Infer the corresponding coalition combination D_1 from the obtained above graph (D_1 should be the set of all maximal cliques of this graph) and construct the carrier, $\widetilde{S}$, of the initial game, $q(N)$, by including only those coalitions from D_1 that fully contain at least one generative coalition of the obtained above unanimity games.

[11]See the reference graphs for the generative coalition combinations of all unanimity games on Figure 4 with generative coalitions in bold.



(a) $q_{(\{1,3\},\{\{2\},\{5\},\{1,3\},\{1,4\}\})}$ (b) $q_{(\{3\},\{\{2\},\{3\},\{5\},\{1,4\}\})}$ (c) $q_{(\{1,4\},\{\{2\},\{3\},\{5\},\{1,4\}\})}$



(d) $q_{(\{1\},\{\{1\},\{2\},\{4\},\{3,5\}\})}$ (e) $q_{(\{1\},\{\{1\},\{2,4\},\{3,5\}\})}$ (f) $q_{(\{3\},\{\{3\},\{1,5\},\{2,4\}\})}$

Figure 4: Unanimity games from the linear decomposition

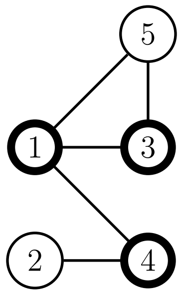


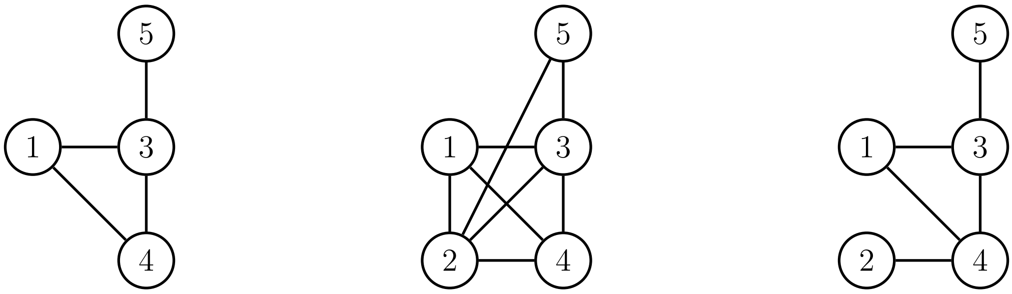
Figure 5: Union of the generative coalition combinations

For our example the corresponding coalition combination is $D_1 = \{\{1,4\},\{2,4\},\{1,3,5\}\}$. *All generative coalitions are:* $\{1\}$, $\{3\}$, $\{1,3\}$ *and* $\{1,4\}$. *So, only players from coalitions* $\{1,4\}$ *and* $\{1,3,5\}$ *from* D_1 *should be in the carrier, because these coalitions contain generative coalitions* $\{1\}$ *and* $\{3\}$ *respectively, while the coalition* $\{2,4\}$ *does not contain any generative coalition. Hence the carrier is* $\widetilde{S} = \{1,3,4,5\}$.

4. Now, in order to construct the new CQFF game on $\widetilde{S}$, we need to be able to give a worth to any integrated coalition on $\widetilde{S}$. So, take any integrated coalition $(S', D') \in IC(\widetilde{S})$.

   For our example we take $(S', D') = (\{1,3,4\},\{\{3,5\},\{1,3,4\}\}) \in IC(\{1,3,4,5\})$.

5. Then construct the coalition combination D on N, such that it satisfies the following. First, its corresponding graph G_D is an upper-graph of one for D', $G_{D'}$. Second, any link



(a) $G_{\{\{3,5\},\{1,3,4\}\}}$ (b) $G_{\{\{2,3,5\},\{1,2,3,4\}\}}$ (c) $G_{\{\{2,4\},\{3,5\},\{1,3,4\}\}}$

Figure 6: Construction of the essence

connecting two i and j that are not both in the carrier (at lest one is not a carrier player) is present in G_D.

From $D' = \{\{3,5\},\{1,3,4\}\}$ *we obtain the coalition combination* $D = \{\{2,3,5\},\{1,2,3,4\}\}$.[12]

6. Now take all unanimity games from the decomposition of the initial game, $q(N)$, whose generating coalition combinations' graphs are sub-graphs of the obtained above graph G_D. Construct a new graph, $G_{\widetilde{D}}$, as a union of the chosen unanimity games' generating coalition combinations' graphs and the graph $G_{D'}$. The corresponding coalition combination, $\widetilde{D}$, will be the essence of D under the initial $q(N)$.

   As we can see, only the first five out of six unanimity games have their generating coalition combinations' graphs as sub-graphs of G_D*, since all graphs on Figure 4 but 4f are sub-graphs of the graph on Figure 6b. So, after the unification of the unanimity games' graphs from Figures 4a–4e and the graph* $G_{D'}$ *from Figure 6a we obtain a new graph* $G_{\widetilde{D}}$ *depicted on Figure 6c. The corresponding coalition combination is* $\widetilde{D} = \{\{2,4\},\{3,5\},\{1,3,4\}\}$. *Hence,* $\widetilde{D} = \{\{2,4\},\{3,5\},\{1,3,4\}\}$ *is the essence of* $D = \{\{2,3,5\},\{1,2,3,4\}\}$ *under* $q(N)$.

7. Finally, we define the worth of the chosen integrated coalition $(S', D') \in IC(\widetilde{S})$ under the new CQFF game $q'(\widetilde{S})$, which is the restriction of the initial game $q(N)$ on its carrier, by setting $q'(S', D') = q(S, \widetilde{D})$.

   For our example we get $q'(\{1,3,4\},\{\{3,5\},\{1,3,4\}\}) = q(\{1,3,4\},\{\{2,4\},\{3,5\},\{1,3,4\}\}) = 1+1+1+1+0.5 = 4.5$. *Since unanimity games with graphs on Figures 4a, 4c, 4d and 4e give all their worth of one to the coalition* $\{1,3,4\}$*, while the unanimity game*

[12]See the reference graphs on Figures 6a and 6b.

with graph on Figure 4b share her worth of one equally between coalitions $\{1,3,4\}$ *and* $\{3,5\}$*. Notice that,* $q(\{2,4\},\{\{2,4\},\{3,5\},\{1,3,4\}\}) = 0$*, since it contains an agent not from the carrier, as Proposition 3 stated.*

Notice that, one also can infer the basis unanimity games and the corresponding coefficients for the obtained restriction $q'(\widetilde{S})$ to the carrier, $\widetilde{S}$, of the initial $q(N)$.

Again suppose that the following unanimity games have not-null coefficients in the decomposition of $q(N)$: $q_{(S_1^1,D_1)},\dots,q_{(S_{l_1}^1,D_1)},\dots,q_{(S_1^k,D_k)},\ \dots,q_{(S_{l_k}^k,D_k)}$. Now from each of those unanimity games on N, $q_{(S_j^i,D_i)}$, construct the corresponding unanimity game, $q_{(S_j^i,D_i')}$, on $\widetilde{S}$, leaving the same generative coalition, and taking the new generative coalition combination D_i' corresponding to the graph $G_{D_i'} \equiv G_{D_i}|_{\widetilde{S}}$, which is a sub-graph of G_{D_i} on $\widetilde{S}$.

For instance, take the unanimity game $q_{(\{1,3\},\{\{2\},\{5\},\{1,3\},\{1,4\}\})}$ *from our example. The corresponding unanimity game on* $\widetilde{S} = \{1,3,4,5\}$ *is* $q_{(\{1,3\},\{\{5\},\{1,3\},\{1,4\}\})}$.

Notice that, two different unanimity games on N, $q_{(S_j^i,D_i)}$ and $q_{(S_j^k,D_k)}$, will have the same corresponding $q_{(S,D')}$ on $\widetilde{S}$, if they have the same generative coalitions, $S_j^i = S_j^k = S$, and their generative coalition combinations' graphs, G_{D_i} and G_{D_k}, have the same sub-graphs $G_{D_i'} = G_{D_k'} = G_{D'}$ on the nodes $\widetilde{S}$.

For example, two unanimity games $q_{(\{1\},\{\{1\},\{2\},\{4\},\{3,5\}\})}$ *and* $q_{(\{1\},\{\{1\},\{2,4\},\{3,5\}\})}$ *have the same corresponding unanimity game on* $\widetilde{S} = \{1,3,4,5\}$*, which is* $q_{(\{1\},\{\{1\},\{4\},\{3,5\}\})}$.

Finally, the restriction $q'(\widetilde{S})$ to the carrier, $\widetilde{S}$, of the initial $q(N)$ is a linear combination of all obtained above corresponding unanimity games on $\widetilde{S}$ with each coefficient for $q_{(S,D')}$, $\alpha_{(S,D')}$, as a sum of the coefficients corresponding to all initial unanimity games with the same corresponding unanimity game on $\widetilde{S}$.

Recall, that we have the following unanimity games in the decomposition of our initial game $q(N)$*, with* $|N| = 5$*, from the example:* $q_{(\{1,3\},\{\{2\},\{5\},\{1,3\},\{1,4\}\})}$, $q_{(\{3\},\{\{2\},\{3\},\{5\},\{1,4\}\})}$, $q_{(\{1,4\},\{\{2\},\{3\},\{5\},\{1,4\}\})}$, $q_{(\{1\},\{\{1\},\{2\},\{4\},\{3,5\}\})}$, $q_{(\{1\},\{\{1\},\{2,4\},\{3,5\}\})}$, $q_{(\{3\},\{\{3\},\{1,5\},\{2,4\}\})}$, *with coefficients* $1,\dots,1$.

Hence, the corresponding unanimity games on $\widetilde{S} = \{1,3,4,5\}$ *are* $q_{(\{1,3\},\{\{5\},\{1,3\},\{1,4\}\})}$, $q_{(\{3\},\{\{3\},\{5\},\{1,4\}\})}$, $q_{(\{1,4\},\{\{3\},\{5\},\{1,4\}\})}$, $q_{(\{1\},\{\{1\},\{4\},\{3,5\}\})}$, $q_{(\{1\},\{\{1\},\{4\},\{3,5\}\})}$ *and* $q_{(\{3\},\{\{3\},\{1,5\},\{4\}\})}$. *So, the restriction* $q'(\widetilde{S})$ *to the carrier of the initial* $q(N)$ *is a linear combination of unanimity games* $q_{(\{1,3\},\{\{5\},\{1,3\},\{1,4\}\})}$, $q_{(\{3\},\{\{3\},\{5\},\{1,4\}\})}$, $q_{(\{1,4\},\{\{3\},\{5\},\{1,4\}\})}$, $q_{(\{1\},\{\{1\},\{4\},\{3,5\}\})}$ *and* $q_{(\{3\},\{\{3\},\{1,5\},\{4\}\})}$ *with coefficients* $1,1,1,1+1=2,1$*, since the forth and the fifth corresponding unanimity games on* $\widetilde{S}$ *are equal.*

3.3 Axioms

Now notice that, for an extension of the classical Shapley value to be applicable, a unanimity CQFF game should not be necessarily superadditive.[13] Since all non-carrier players will be irrelevant between entering a coalition or not, while the total worth of this coalition stays zero. And because, for the carrier players to obtain any worth there should appear at least the essence of the grand coalition combination (or its upper-graph), $\{N\}$, and from Proposition 3 we know that, any coalition from this essence containing at least one non-carrier player receives zero worth. Hence, at least the essence of the grand coalition combination will form, which is enough for distribution of all worth of one (of the grand coalition under a unanimity game) between the carrier players. Simply speaking, in case of any unanimity CQFF game, agents will for sure continue connecting and forming new coalitions till some not-null worth appears, which is, fortunately, the worth of the grand coalition.[14]

Let us denote the *set of all superadditive and unanimity CQFF games on* N as $\mathcal{Q}^N$. Then we define a *value of CQFF games* as a mapping $\varphi : \mathcal{Q}^N \to \mathbb{R}^n$.[15] So, the domain of a *value of CQFF games* is $dom(\varphi) = \mathcal{Q}^N$.

At last, we have done all preliminary work. Now, for the extension of the value, proposed by Myerson, we will need the following three axioms.

Efficiency axiom. For each CQFF game $q(N) \in \mathcal{Q}^N$, if $\widetilde{S}$ is the carrier of $q(N)$, then $\sum\limits_{i \in S} \varphi_i(q(N)) = q(N, \{N\})$ for any coalition $S \supseteq \widetilde{S}$.[16]

If a permutation only swaps two players i, j leaving all the others on the same places we denote it as σ_{ij}.

[13]We need this, since the defined symmetric unanimity games are not superadditive. Consider the unanimity game on 4 players $q_{(\{1\},\{\{1\},\{2\},\{3\},\{4\}\})} \equiv q$. Take the coalition combination $D = \{\{1,2\},\{1,3\},\{1,4\}\}$. We now have the following worths: $q(\{1,2\}, D) = q(\{1,3\}, D) = q(\{1,4\}, D) = 1/3$. Now merge two coalitions $\{1,2\}$ and $\{1,3\}$, as it is shown in the definition of superadditivity, and obtain the new coalition combination $D' = \{\{1,2,3\},\{1,4\}\}$ with worths $q(\{1,2,3\}, D') = q(\{1,4\}, D') = 1/2$. Finally, if q is superadditive, than we should have $q(\{1,2\}, D) + q(\{1,3\}, D) \leq q(\{1,2,3\}, D')$, but $2/3 \not\leq 1/2$.

[14]The same arguments are applicable to any type of the unanimity CQFF game (not necessarily symmetric), i.e. CQFF game with only one integrated coalition providing all the worth.

[15]As a reference point for the extension of the Shapley value on CQFF games we choose the value, proposed in Myerson (1977), for PFF games. All existing by now extensions for PFF games are described in Macho-Stadler et al. (2017).

[16]By this we require all non-carrier players to receive zero value. So, the carrier of the CQFF game is an analog to minimal carriers of CFF or PFF games, which are defined in the Appendix A.

Carrier symmetry axiom. For each CQFF game $q(N) \in \mathcal{Q}^N$, two players from its carrier $i, j \in \widetilde{S}$ receive the same value, $\varphi_i(q(N)) = \varphi_j(q(N))$, if the following holds. The restricted to the carrier $\widetilde{S}$ game, $q'(\widetilde{S})$, stays the same under permutation σ_{ij}, $\sigma_{ij} q'(\widetilde{S}) = q'(\widetilde{S})$, where $(\sigma q)(S, D) \equiv q(\sigma S, \sigma D)$.

Linearity axiom. For any pair of CQFF games $q(N), q'(N) \in \mathcal{Q}^N$, the following holds: $\varphi_i(q(N) + q'(N)) = \varphi_i(q(N)) + \varphi_i(q'(N))$ for any agent $i \in N$.

3.4 Value

Let us define the following Shapley value φ^{Sp}. For each unanimity game, $q_{(S',D')}(N) \in \mathcal{Q}^N$, and for each player $i \in N$ set

$$\varphi_i^{Sp}\left(q_{(S',D')}\right) = \begin{cases} 1/s & \text{if } i \in S', \text{ and } |S'| = s \\ 0 & \text{otherwise.} \end{cases}$$

Since any non-unanimity CQFF game $q(N) \in \mathcal{Q}^N$ can be expressed as a linear combination of unanimity games uniquely, $q(N) = \sum\limits_{(S',D') \in IC(N)} \alpha_{(S',D')} q_{(S',D')}$, we define the Shapley value φ^{Sp} for any CQFF game as the corresponding linear combination of values for unanimity games as follows:

$$\varphi^{Sp}(q(N)) = \sum_{(S',D') \in IC(N)} \alpha_{(S',D')} \varphi^{Sp}\left(q_{(S',D')}\right).$$

Notice that, if $q'(\widetilde{S})$ is the restriction of $q(N)$ on its carrier $\widetilde{S}$, then $\varphi_i^{Sp}\left(q'(\widetilde{S})\right) = \varphi_i^{Sp}(q(N))$, for any carrier player $i \in \widetilde{S}$.

Now we can use the introduced above three axioms to prove the following theorem.

Theorem. *There exists a unique value for CQFF games, φ^{Sp}, satisfying efficiency, carrier symmetry and linearity.*

Proof. Follows straightly from Lemmas 1 and 2 below. □

Lemma 1. *The value φ^{Sp} satisfies efficiency, carrier symmetry and linearity.*

Proof. Let the CQFF game $q(N) \in \mathcal{Q}^N$ be given. Firstly, since the union of all generative coalitions of the unanimity games in the decomposition of $q(N)$ is a subset of the carrier of $q(N)$, all players not from the carrier will receive zero value. So, the value φ^{Sp} satisfies efficiency.

Secondly, the value φ^{Sp} satisfies linearity by definition.

Now let us turn to carrier symmetry.

Take the restriction of the initial game $q(N)$ on its carrier $\widetilde{S}$, $q'(\widetilde{S})$. Let players i, j be from the carrier $\widetilde{S}$ and assume that $\sigma_{ij}q'(\widetilde{S}) = q'(\widetilde{S})$. Consider a partition of the set of all possible unanimity games on the carrier players $\widetilde{S}$ into either pairs $\{q_{(S',D')}, q_{(\sigma_{ij}S',\sigma_{ij}D')}\}$ or singletons $\{q_{(S',D')}\}$, for which $q_{(S',D')} = q_{(\sigma_{ij}S',\sigma_{ij}D')}$. Recall that our restricted game can be **uniquely** decomposed into a linear combination of unanimity games. Hence, the decompositions of $q'(\widetilde{S})$ and $\sigma_{ij}q'(\widetilde{S})$ should be the same, since they are equal.

Now notice that, after permuting players i and j in $q'(\widetilde{S})$ the coefficients in the decomposition swap inside each pair $\{q_{(S',D')}, q_{(\sigma_{ij}S',\sigma_{ij}D')}\}$. If we denote coefficients for the game $q'(\widetilde{S})$ as α, and for the game $\sigma_{ij}q'(\widetilde{S})$ as α', we have: $\alpha_{(S',D')} = \alpha'_{(\sigma_{ij}S',\sigma_{ij}D')}$ for any $(S', D') \in IC(\widetilde{S})$. But since the decompositions of $q'(\widetilde{S})$ and $\sigma_{ij}q'(\widetilde{S})$ are the same, we have that $\alpha_{(S',D')} = \alpha'_{(S',D')}$ for any $(S', D') \in IC(\widetilde{S})$.

Finally, from above we get that equality $\alpha_{(S',D')} = \alpha'_{(S',D')} = \alpha_{(\sigma_{ij}S',\sigma_{ij}D')}$ holds for any $(S', D') \in IC(\widetilde{S})$. Hence players i and j have to receive the same value under the restricted game $q'(\widetilde{S})$. Thus, $\varphi_i(q') = \varphi_j(q')$. Therefore they will receive the same value under the initial game $q(N)$, $\varphi_i(q) = \varphi_j(q)$, since for any carrier player $i \in \widetilde{S}$ her value is the same under the initial and the restricted games (by construction of the latter one): $\varphi_i(q') = \varphi_i(q)$ for all $i \in \widetilde{S}$. While all non-carrier players receive zero values.

Hence, the value φ^{Sp} satisfies carrier symmetry. $\square$

Lemma 2. *There is at most one value, φ^{Sp}, which satisfies efficiency, carrier symmetry and linearity.*

Proof. Take any unanimity game $q_{(S',D')} \in \mathcal{Q}^N$ on N. Firstly, S' is the carrier of $q_{(S',D')}$. Secondly, the restriction of $q_{(S',D')}$ to the carrier is the unanimity game $q_{(S',\{S'\})}$, which gives worth of one to the integrated coalition $(S', \{S'\}) \in IC(S')$ and zero to all other integrated coalitions from $IC(S')$.

The efficiency axiom requires all the worth of one to be distributed only between players in S' of the unanimity game $q_{(S',D')}$.

The carrier symmetry axiom requires each player from S' of the unanimity game $q_{(S',D')}$ to have the same value. Since, by permuting any two agents $\{i, j\} \subseteq S'$ the obtained game $\sigma_{ij}q_{(S',\{S'\})}$ will also give worth of one to the integrated coalition $(S', \{S'\}) \in IC(S')$ and zero to all other integrated coalitions from $IC(S')$, as $q_{(S',\{S'\})}$ did.

Hence, any unanimity game $q_{(S',D')}$ will raise the above defined value $\varphi^{Sp}\left(q_{(S',D')}\right)$.

Finally, since any CQFF game $q(N)$ can be uniquely decomposed into a linear combination of unanimity games, the linearity axiom requires that any game $q(N)$ should raise the unique defined value $\varphi^{Sp}\left(q(N)\right)$. $\square$

Notice that, under a PFF game setting the value φ^{Sp} is equal to the value, proposed by Myerson.[17]

3.5 Independence of the axioms

In this section we prove that the proposed axioms are independent. In other words, we show that no pair of these three axioms imply the third one. For each pair of axioms we present a value for a CQFF game satisfying both axioms from this pair, but not the other one.

Given an arbitrary CQFF game $q(N) \in \mathcal{Q}^N$ with the carrier $\widetilde{S}$ and the decomposition

$$q(N) = \sum_{(S',D') \in IC(N)} \alpha_{(S',D')} q_{(S',D')}$$

these values are as follows.

1. *Efficiency and carrier symmetry*

$$\varphi_i^1\left(q(N)\right) = \begin{cases} q(N,\{N\})/s & \text{if } i \in \widetilde{S}, \text{ and } |\widetilde{S}| = s \\ 0 & \text{otherwise.} \end{cases}$$

The proposed value, φ^1, clearly satisfies efficiency and carrier symmetry, since it equally distributes all the worth across agents only from the carrier.

Now consider a superadditive game on two agents with the following decomposition to the unanimity games: $q(N) = q_{(\{1\},\{\{1\},\{2\}\})} + q_{(\{1,2\},\{\{1,2\}\})}$. Obviously, the carrier of $q(N)$ is $\widetilde{S} = \{1,2\}$. Under the first unanimity game we have the following value: $\varphi^1\left(q_{(\{1\},\{\{1\},\{2\}\})}\right) = (1,0)$. While, under the second unanimity game we have the following value: $\varphi^1\left(q_{(\{1,2\},\{\{1,2\}\})}\right) = (1/2,1/2)$. So, linearity requires the following value for the game $q(N)$: $(1,0) + (1/2,1/2) = (3/2,1/2)$. However, the proposed value raises $\varphi^1\left(q(N)\right) = (1,1) \neq (3/2,1/2)$. Hence, the value φ^1 does not satisfy linearity.

2. *Efficiency and linearity*

For unanimity games we have

$$\varphi_i^2\left(q_{(S',D')}\right) = \begin{cases} 1 & \text{if } i \in S', i = \min\{j \mid j \in S'\} \\ 0 & \text{otherwise.} \end{cases}$$

[17] See the Appendix Section A.2 for a characterization of the value, proposed by Myerson.

Then we have

$$\varphi^2\left(q(N)\right) = \sum_{(S',D')\in IC(N)} \alpha_{(S',D')}\varphi^2\left(q_{(S',D')}\right).$$

The proposed value, φ^2, satisfies efficiency, since it distributes all the worth across agents only from the carrier. Also, it satisfies] linearity by construction.

Now consider the following unanimity game on two players, $q_{(\{1,2\},\{\{1,2\}\})}$. It will raise the following value: $\varphi^2\left(q_{(\{1,2\},\{\{1,2\}\})}\right) = (1,0)$, since $1 = \min\{j \mid j \in \{1,2\}\}$. But notice that, firstly, the unanimity game is the restriction to the carrier of itself, and secondly, that it stays the same under permutation σ_{12}, $\sigma_{12}q_{(\{1,2\},\{\{1,2\}\})} = q_{(\{1,2\},\{\{1,2\}\})}$. So, carrier symmetry requires equal values for both players $(1/2,1/2) \neq (1,0)$. Hence, the value φ^2 does not satisfy carrier symmetry.

3. *Carrier symmetry and linearity*

For unanimity games we have

$$\varphi_i^3\left(q_{(S',D')}\right) = 1/|N| \text{ for all } i \in N$$

Then we have

$$\varphi^3\left(q(N)\right) = \sum_{(S',D')\in IC(N)} \alpha_{(S',D')}\varphi^3\left(q_{(S',D')}\right).$$

The proposed value, φ^3, satisfies carrier symmetry, since it equally distributes all the worth across all agents. Also, it satisfies linearity by construction.

Now consider the following unanimity game on two players, $q_{(\{1\},\{\{1\},\{2\}\})}$. It will raise the following value: $\varphi^3\left(q_{(\{1\},\{\{1\},\{2\}\})}\right) = (1/2,1/2)$. Notice that, agent 2 is not in the carrier of this unanimity game, however, she receives not-null value under φ^3. Hence, the value φ^3 does not satisfy efficiency.

So, the presented above examples prove that the proposed three axioms are indeed independent.

4 Domain cardinality of a CQFF game

Unfortunately, in terms of the domain of a game, the transition from PFF to CQFF settings gives birth to even more severe curse of dimensionality, then it was for the transition from CFF to PFF settings. If we take n divers players, then we will have 2^n coalitions (the domain size of a CFF game), $B(n)$ ($n-$th Bell number (Bell, 1934)) partitions (proxy for the domain size of a PFF game), and $2^{n(n-1)/2}$ coalition combinations (proxy for the domain size of a CQFF game). This fact leads to a computational complexity of the constructed Shapley value for CQFF games. However, there is a way to simplify the calculations by restricting the domain of a CQFF game through the following logic.

Since one player generally should divide her time among all coalitions she participates in, it would be interesting to find how the *number of different coalition combinations (graphs) on n divers agents, such that each agent participates in no more than l coalitions*, changes if l changes.

We focus firstly on the specific case, where a coalition cannot contain more than two players.

Proposition 4. *The maximal cardinality of a coalition combination, where coalitions cannot contain more than two agents, on $n \geq 2$ divers agents is*

$$\xi_2(n) = \begin{cases} \lfloor \frac{n}{2} \rfloor \left(\lfloor \frac{n}{2} \rfloor + \mathbb{I}(n \textit{ is odd}) \right) & \textit{if } n \geq 4 \\ n & \textit{if } n \in \{2, 3\}, \end{cases}$$

where $\mathbb{I}(x)$ is an index function that takes value 1, if condition x is true and 0, otherwise, and $\lfloor y \rfloor$ is a rounding down of a real number y.

Proof. Since there cannot be a coalition containing three or more agents, the graph representation of any suitable coalition combination will not have triangles. Also, under this restriction any edge (link) of the graph will represent a coalition, containing agents connected by this edge.

The result follows straightly from the Turán's theorem (Turan, 1941), that implies that n-vertex triangle-free graph with the maximum number of edges is a complete bipartite graph (its set of vertices can be divided into two disjoint subsets A and B, such that each vertex from A is linked to anyone from B and vice-versa, and no two vertices from the same subset are linked) in which the numbers of vertices on each side of the bipartition are as equal as possible. Obviously, there is another opportunity for making coalitions: leaving some nodes alone. However, destroying all edges of any node of the described above graph will decrease the total number of coalitions by at least $(\lfloor n/2 \rfloor - 1) > 0$, if $n \geq 4$ (destroying either $\lfloor n/2 \rfloor$ or $(\lfloor n/2 \rfloor + 1)$ coalitions, depending on which side of the bipartition our node was, and creating new coalition-singleton).

Hence, if $n \geq 4$ is even we will have $(n/2)^2$ edges, and if $n = a + 1$ is odd we will have

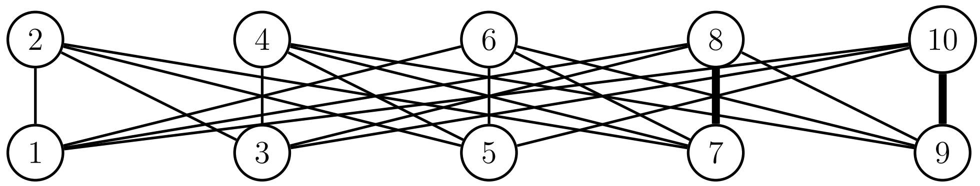


Figure 7

$[(a/2)^2 + (a/2)]$ edges.

As for $n \in \{2, 3\}$, obviously $\xi_2(n) = n$, since we will not need any links. Because for these cases only coalitions-singletons makes the coalition combination with the biggest cardinality possible. □

Corollary 1. *The maximal cardinality of a coalition combination, where coalitions cannot contain more than two agents, on $n \geq 2$ divers agents, such that each agent participates in no more than $l > 1$ coalitions is*

$$\xi_2(n, l) = \begin{cases} \xi_2(n) & \text{if } l > \left\lfloor \frac{n}{2} \right\rfloor \\ \frac{1}{2}[nl - \mathbb{I}(l \text{ is odd})\mathbb{I}(n \text{ is odd})] & \text{if } l \leq \left\lfloor \frac{n}{2} \right\rfloor \end{cases}.$$

Proof. Case for $l > \lfloor n/2 \rfloor$ follows straightly from Proposition 4.

If $l \leq \lfloor n/2 \rfloor$ and n is even, we can simply obtain bipartite graph (not necessarily complete) with $n/2$ vertices on each side of the bipartition, where each node has vertexity exactly l. See example for $n = 10$ and $l = 4$ on Figure 7. Hence overall the number of links is $l(n/2)$.

However, if $n = a + 1$ is odd, after adding an additional node to the bipartite graph with a nodes, where each one has vertexity exactly l, we will need to include this node into the graph by destroying existing $\lfloor l/2 \rfloor$ links, that had no common nodes, and then replace each of them by path consisting of two edges connecting the former pair of nodes with the new one.[18]

By this we will make additional node's vertexity l if l is even, and $(l - 1)$ – otherwise, without affecting vertexities of the existing nodes. □

As for the general case, where we do not restrict the number of agents in coalitions, we have the following result:

Proposition 5. *The maximal cardinality of a coalition combination on $n \geq 2$ divers*

[18]See an example of such transformation from graph on Figure 7 to one on Figure 8, where we destroyed links connecting $\{8, 7\}$ and $\{9, 10\}$ and introduced $\{8, 11\}$ and $\{11, 7\}$, and $\{10, 11\}$ and $\{11, 9\}$.

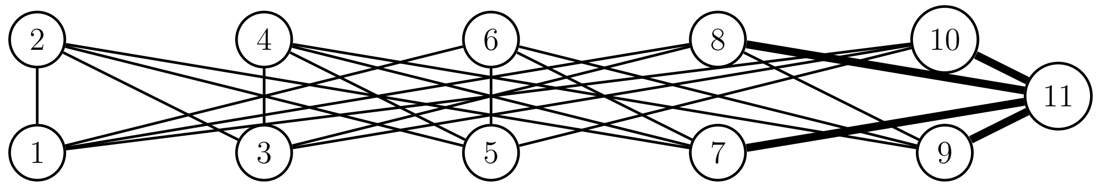


Figure 8

agents is

$$\xi(n) = \begin{cases} 3^{n/3} & \text{if } n \equiv 0 \pmod 3 \\ 4 \cdot 3^{\lfloor n/3 \rfloor - 1} & \text{if } n \equiv 1 \pmod 3 \\ 2 \cdot 3^{\lfloor n/3 \rfloor} & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Proof. See Moon and Moser (1965) for the proof. □

The main idea is that for the graph with maximal number of maximal cliques on n labeled nodes we can construct another graph with the same number of maximal cliques (coalitions) with the following structure. The set of all its n nodes can be partitioned into k subsets with $j_1, \ldots, j_k$ nodes, such that each node is connected to all nodes from the other subsets and is not connected to any node from her own subset.

Hence, the total amount of maximal cliques is a product $\left[\prod_{i=1}^{k} j_i\right]$, which is maximized when all j_i are equal to 3, except one. This last one should be one of three: 2, 3 or 4, depending on n.

Now we can denote the *number of different coalition combinations (different graphs) with cardinality k on n divers agents, such that each agent participates in no more than l coalitions (maximal cliques)*, as $M(n, k, l)$ (for positive n, k, l).

Corollary 2. *The above defined number satisfies the following:*

1. *$M(n, k, l) = 0$, if $k > \xi(n, l)$, where $\xi(n, l)$ is the maximal cardinality of a coalition combination on $n \geq 2$ divers agents, such that each agent participates in no more than $l > 1$ coalitions;*

2. *$M(n, k, 1)$ is the number of ways to partition a set of n different objects into k non-empty subsets (Stirling number of the second kind).*

Proof. The first equality follows straightly from Corollary 1.

As for the second one, if $l = 1$ we do not have multiple memberships, hence we are considering only partitions. □

Hence, we will obtain the *number of partitions of a set of size n (Bell numbers (Bell, 1934))* by summing $M(n, k, 1)$ for all k. Let us denote in general

$$M(n,l) \equiv \sum_{k=1}^{\xi(n,l)} M(n,k,l).$$

The question now is the following: what effect on $M(n,l)$ will have the change of l, keeping n fixed?

From Proposition 5 we obtain the following.

Corollary 3. *For any fixed $n \geq 2$, $M(n,l) \leq 2^{n(n-1)/2}$, where inequality is strict for $l < \xi(n-1)$.*

Proof. The non-strict inequality part is obvious, since, for fixed n, number $M(n,l)$ (the number of different coalition combinations (different graphs) on n divers agents, such that each agent participates in no more than l coalitions (maximal cliques)) cannot exceed the total number of different graphs on n divers nodes, which is exactly $2^{n(n-1)/2}$.

As for the strict inequality case, we need to answer the following question. What is the maximal number of coalitions that one agent can participate in, if the total number of agents is fixed to n? Since only till this boundary the value of l will affect $M(n,l)$.

Notice, that if we have a graph on n agents, where one agent a participates in c coalitions, we can do the following procedure. Firstly, we add a new agent a_1 and connect her to all $(n-1)$ agents, excluding a. Now we get at least $2c$ coalitions in total and this additional agent a_1 participates in c coalitions. Secondly, we add second new agent a_2 and again connect her to all $(n-1)$ agents, excluding a and a_1. Now we get at least $3c$ coalitions in total and this second additional agent a_2 participates in c coalitions. So, finally, we obtain a graph on $(n+2)$ nodes with at least $3c$ coalitions.

Also notice, that for any n the following equality holds $3\xi(n) = \xi(n+3)$:

1. If $n \equiv 0 \pmod 3$, then $3\xi(n) = 3 \cdot 3^{n/3} = 3^{(n+3)/3} = \xi(n+3)$;

2. If $n \equiv 1 \pmod 3$, then $3\xi(n) = 3 \cdot 4 \cdot 3^{\lfloor n/3 \rfloor - 1} = 4 \cdot 3^{\lfloor (n+3)/3 \rfloor - 1} = \xi(n+3)$;

3. If $n \equiv 2 \pmod 3$, then $3\xi(n) = 3 \cdot 2 \cdot 3^{\lfloor n/3 \rfloor} = 2 \cdot 3^{\lfloor (n+3)/3 \rfloor} = \xi(n+3)$.

l \ n	1	2	3	4	5	6	7	8
1	1	**2**	**5**	15	52	203	877	4 140
$2 = \xi(2)$	-	-	8	**60**	**694**	10 790	210 124	4 963 734
$3 = \xi(3)$	-	-	-	64	1 004	**28 490**	**1 314 174**	90 218 858
$4 = \xi(4)$	-	-	-	-	1 024	32 180	1 878 129	**187 961 713**
5	-	-	-	-	-	32 708	2 053 311	242 236 017
$6 = \xi(5)$	-	-	-	-	-	32 768	2 091 202	261 767 613
7	-	-	-	-	-	-	2 096 242	266 977 517
8	-	-	-	-	-	-	2 097 082	268 194 992
$9 = \xi(6)$	-	-	-	-	-	-	2 097 152	268 392 112
10	-	-	-	-	-	-	-	268 430 976
11	-	-	-	-	-	-	-	268 434 336
$12 = \xi(7)$	-	-	-	-	-	-	-	268 435 456

Table 1: Values of $M(n, l)$

We claim that the answer for the question above is $\xi(n-1)$. Let us prove it by contrary. Suppose, that the answer is $(\xi(n-1)+q) > \xi(n-1)$, so we can have a graph on n agents, in which one participates in $(\xi(n-1)+q)$ coalitions. Applying the described above procedure we obtain a graph on $(n+2)$ nodes with at least $3(\xi(n-1)+q) > 3\xi(n-1) = \xi(n-1+3) = \xi(n+2)$ coalitions. Contradiction to Proposition 5.

Now, we construct an example of a graph on n agents, such that there exist one agent participating in $\xi(n-1)$ coalitions. We keep our one agent alone and let other $(n-1)$ agents organize themselves into $\xi(n-1)$ coalitions as we saw in the proof of Proposition 5, and then connect this agent with all the others. Hence, she will participate in exactly $\xi(n-1)$ coalitions, which is the answer to our question above. $\square$

With the use of MATLAB we obtained Table 1 of values of $M(n, l)$ for $n \in \{1, \ldots, 8\}$.

Sign '-' means, that the number is the same as above. The first row contains exactly the Bell numbers. Also, for each $n \geq 2$ the value of $M(n, l)$ stops changing and stays equal to $2^{n(n-1)/2}$ after l exceeds $\xi(n-1)$, as Corollary 3 stated. The numbers in bold correspond to the $M(n, \lfloor n/2 \rfloor)$.

Moreover, using Corollary 1 we realize that, if we have a constraint $l > 1$ for the number of coalitions any agent can participate in, and if we can construct the graph in which each agent participates in exactly l coalitions (or each except one agent, which participates in $(l-1)$ coalitions, if l and n are both odd), then the maximal possible number of coalitions in this graph will increase if the maximal allowed cardinality of a coalition decreases. And also this graph can potentially have the biggest possible number of coalitions for fixed n. Hence, the best possible graph will be the one from the proof of Corollary 1, which has $\xi_2(n, l)$ coalitions. So, for $l \le \lfloor n/2 \rfloor$: $\xi(n, l) = \xi_2(n, l)$.

Hence, for any $n > 2$ we have $M(n, \lfloor n/2 \rfloor) > (1/2) \cdot 2^{n(n-1)/2} = (1/2) \cdot M(n, \xi(n-1))$ (from Table 1) and, for $l \le \lfloor n/2 \rfloor$, $\xi(n, l) = \xi_2(n, l)$. Proceeding from the foregoing, we can formulate the following surmise.

Conjecture. *For any fixed $n > 2$, by restricting the set of all graphs (coalition combinations) to graphs that satisfy the condition $l \le \lfloor n/2 \rfloor$, we will get the majority of graphs ($\approx$ 60–70%). Moreover, for this restricted set we will have $\xi(n, l) = \xi_2(n, l)$.*

As the final outcome, if, for instance we focus on the restriction $l \le 2$, then for any fixed number of players, n, we will have only $M(n, 2)$ coalition combinations relevant to any CQFF game. And, as the result, the new domain of a CQFF game will consist of only those integrated coalitions, which have one out of those $M(n, 2)$ coalition combinations as the second element. From Table 1 we see that such a restriction will significantly reduce the domain of a CQFF game. And, as the result, simplify the derivation of the proposed Shapley value.

5 Final remarks

It this paper, we introduced the new concept of the clique function form (CQFF) games, which describes the cooperative games with explicit externalities and multiple membership. The core idea for this concept is to adopt a graphical representation to the cooperative setting through a different logic, than in the communication games.

In the communication games, a coalition corresponds to a connected component of the graph, with players as vertices and friendships as edges. Since, if one player can reach another one through the chain of the acquainted agents, then they should be the members of one coalition. In such approach it turns out that, two members from one coalition may not have an opportunity to directly communicate. Also, no matter how we construct the graph, any agent will always be in exactly one coalition.

However, under the CQFF game framework, we treat a maximal clique of a graph (any maximal fully interlinked sub-graph) as a coalition. Now, two described drawbacks no longer exist, since any agent can directly interact with any of her co-members, and, in addition, she can participate in multiple coalitions.

The main reason to study such setting is that multiple membership and externalities could be found in lots of interesting environments. For instance, CQFF games could be used as an introducing externalities application to the assignment games (Shapley and Shubik, 1971) by using a bipartite graph. Therefore, the proposed approach could be of interest to researchers looking for solution concepts in such environments.

One important feature of the constructed value is that, for any chosen basis of unanimity games (not necessarily symmetric, but natural: for any given unanimity game, under any coalition combination, the sum of all the worths of coalitions, containing the generative coalition, adapts to one) we will obtain a new value for a given game. This value will be also fully justified, since all provided in this paper notions and proofs, that appeared after we chose symmetric unanimity games as basis for the CQFF games space, did not use the symmetry of these unanimity games.[19] This raises a similar concept to the *weighted Shapley value* (Shapley, 1953b). Because, if basis unanimity games are not symmetric, it means that, carrier players treat dummy-players differently, for instance, according to initially given weights.

Another fascinating question to study was the size of a set of all coalition combinations under different upper bounds, l, for number of coalitions a player could participate in. For instance, imagine that it is stated that, any agent could be a member of at most three coalitions. How many relevant coalition combinations, out of all possible $2^{n(n-1)/2}$, are left now? Although we did not find the general solution to this problem; for a fixed number of agents, n, we obtained the threshold level of the upper bound l, above which the number of interest stays constant. Withal, we came to a conclusion that, by setting the upper bound to $l = \lfloor n/2 \rfloor$ we will keep around 60–70% of all coalition combinations.

During our further research we contemplate to study a family of values produced by all possible bases of natural unanimity games. Also, we plan to derive an extension of the core for CQFF games. Moreover, it would be interesting and challenging to develop a stable link formation process (Carayol and Roux (2009), Aumann and Myerson (2003)) or even to perform a computational study like in Skibski et al. (2015).

[19] Generally speaking, all results will hold for any chosen basis, non necessarily natural. So, set of all possible values of an CQFF game could be further extended.

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A Values for CFF and PFF games

A.1 Shapley value (CFF games)

Consider the definition of the characteristic function form (CFF) games from Section 2.

Definition. A CFF game $v(N)$ is *superadditive* if for any two disjoint coalitions S and S' the following holds: $v(S \cup S') \geq v(S) + v(S')$. We denote the set of all superadditive CFF games on N as $\mathcal{V}^N$.

Definition. A *value φ for superadditive CFF games* is a mapping $\varphi(v) : \mathcal{V}^N \to \mathbb{R}^n$.

Definition. Given a CFF game $v(N)$ we call the set of players $\widetilde{S} \subseteq N$ a *carrier of* $v(N)$ if for any coalition $S \subseteq N$ the following holds: $v(S \cap \widetilde{S}) = v(S)$.[20]

Definition. For any coalition $S' \subseteq N$ we have the corresponding *unanimity game*

$$v_{S'}(S) = \begin{cases} 1 & \text{if } S \supseteq S' \\ 0 & \text{otherwise.} \end{cases}$$

The Shapley value φ^{Sh} is characterized by the following three axioms.

Symmetry axiom. For each given superadditive CFF game $v(N)$, for each permutation π on N the following holds: $\varphi^{Sh}_{\pi i}(\pi v) = \varphi^{Sh}_i(v)$ for any $i \in N$, where $\pi v(S) \equiv v(\pi S)$ for any coalition $S \subseteq N$.

Efficiency axiom. For each given superadditive CFF game $v(N)$, if $\widetilde{S} \subseteq N$ is a carrier of $v(N)$, the following holds: $\sum_{i \in \widetilde{S}} \varphi^{Sh}_i(v) = v(N)$.

Linearity axiom. For any two superadditive CFF games $v(N)$ and $v'(N)$ the following holds: $\varphi^{Sh}(v) + \varphi^{Sh}(v') = \varphi^{Sh}(v + v')$.

Theorem. *A unique value function φ^{Sh} exists satisfying symmetry, efficiency and linearity, for any superadditive CFF game $v(N)$; it is given by the formula*

$$\varphi^{Sh}_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!\,(n - |S| - 1)!}{n!} \left(v(S \cup \{i\}) - v(S)\right) \text{ for any } i \in N.$$

[20] Notice that the grand coalition N is also always a carrier of a CFF game. But there is only one carrier that is included in all others. In our paper we refer to is as a *minimal carrier of a CFF game*.

A.2 Value, proposed by Myerson, (PFF games)

Consider the definition of the partition function form (PFF) games from Section 2.

Definition. A PFF game $w(N)$ is *superadditive* if for any two embedded coalitions (S,P) and (S',P) the following holds: $w(S \cup S', P \backslash (\{S\} \cup \{S'\}) \cup \{S \cup S'\}) \geq w(S,P) + w(S',P)$. We denote the set of all superadditive PFF games on N as $\mathcal{P}^N$.

Definition. A *value* φ *for superadditive PFF games* is a mapping $\varphi(w) : \mathcal{P}^N \to \mathbb{R}^n$.

Definition. For any two partitions P and P' we define an *intersection of partitions* as $P \wedge P' = \{S \cap S' \,|\, S \in P, S' \in P', S \cap S' \neq \emptyset\}$. For two embedded coalitions (S,P) and (S',P') we write $(S,P) \gg (S',P')$ if $S' \subseteq S$ and $P \wedge P' = P'$.

Definition. Given a PFF game $w(N)$ we call the set of players $\widetilde{S} \subseteq N$ a *carrier of* $w(N)$ if for any embedded coalition $(S,P) \in EC(N)$ the following holds: $w(S \cap \widetilde{S}, P \wedge \{\widetilde{S}, N \backslash \widetilde{S}\}) = w(S,P)$.[21]

Definition. For any embedded coalition $(S',P') \in EC(N)$ we have the corresponding *unanimity game*

$$w_{(S',P')}(S,P) = \begin{cases} 1 & \text{if } (S,P) \gg (S',P') \\ 0 & \text{otherwise.} \end{cases}$$

The value, proposed by Myerson, φ^M is characterized by the following three axioms.

Symmetry axiom. For each given superadditive PFF game $w(N)$, for each permutation π on N the following holds: $\varphi^M_{\pi i}(\pi w) = \varphi^M_i(w)$ for any player $i \in N$, where $\pi w(S) \equiv w(\pi S, \pi P)$ for any embedded coalition $(S,P) \in EC(N)$.

Efficiency axiom. For each given superadditive PFF game $w(N)$, if $\widetilde{S} \subseteq N$ is a carrier of $w(N)$, the following holds: $\sum_{i \in \widetilde{S}} \varphi^M_i(w) = w(N, \{N\})$.

Linearity axiom. For any two superadditive PFF games $w(N)$ and $w'(N)$ the following holds: $\varphi^M(w) + \varphi^M(w') = \varphi^M(w + w')$.

[21] Notice that the grand coalition N is also always a carrier of a PFF game. But there is only one carrier that is included in all others. In our paper we refer to is as a *minimal carrier of a PFF game*.

The value, proposed by Myerson, φ^M is defined as follows. For each unanimity game, $w_{(S',P')}(N) \in \mathcal{P}^N$, and for each player $i \in N$ set

$$\varphi_i^M \left(w_{(S',P')}\right) = \begin{cases} 1/s & \text{if } i \in S',\ \text{and } |S'| = s \\ 0 & \text{otherwise.} \end{cases}$$

Since any PFF game $w(N) \in \mathcal{P}^N$ can be expressed as a linear combination of the unanimity games uniquely, $w(N) = \sum\limits_{(S',P') \in EC(N)} \alpha_{(S',P')} w_{(S',P')}$, the value φ^M for any PFF game is defined as the corresponding linear combination of the values for the unanimity games as follows:

$$\varphi^M \left(w(N)\right) = \sum_{(S',P') \in EC(N)} \alpha_{(S',P')} \varphi^M \left(w_{(S',P')}\right)$$

Theorem. *There exists a unique value for PFF games, φ^M, satisfying efficiency, symmetry and linearity.*