Maple-assisted proof of recurrence for A296596

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There are $2^{10} = 1024$ configurations for a 2×5 sub-array, but not all can arise.

We encode these configurations as lists in the order $\begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ x_6 & x_7 & x_8 & x_9 & x_{10} \end{bmatrix}$. Each of $x_2, x_3, x_4, x_7, x_8, x_9$ is adjacent to 4 other entries, and those 5 can't all be 1's.

```
> goodconfig:= proc(x)
     x[2]+x[1]+x[3]+x[6]+x[7]<=4
     and x[3]+x[2]+x[4]+x[7]+x[8]<=4
     and x[4]+x[3]+x[5]+x[8]+x[9]<=4
     and x[7]+x[2]+x[3]+x[6]+x[8]<=4
     and x[8]+x[3]+x[4]+x[7]+x[9]<=4
     and x[9]+x[4]+x[5]+x[8]+x[10]<=4
  end proc:
  Configs:= select(goodconfig, [seq(convert(2^10+i,base,2)[1..10],
  i=0..2^10-1)]):
  nops(Configs);
```

$$912 \tag{1}$$

There are 912 allowed configurations.

Consider the 912×912 transition matrix T with entries $T_{ij} = 1$ if the first two rows of a 3×5 sub-array could be in configuration i while the last two rows are in configuration j, and 0 otherwise. The following code computes it.

```
> Compatible:= proc(i,j) local Xi,Xj,k;
   Xi:= Configs[i]; Xj:= Configs[j];
   if Xi[6..10] <> Xj[1..5] then return 0 fi;
   if Xj[1]=1 and not member(Xi[1]+Xi[2]+Xj[2]+Xj[6], {1,2,3}) then
  return 0 fi;
   if Xj[2]=1 and not member(Xi[2]+Xi[3]+Xj[1]+Xj[3]+Xj[6]+Xj[7],
  {1,2,3}) then return 0 fi;
   if Xj[3]=1 and not member(Xi[3]+Xi[4]+Xj[2]+Xj[4]+Xj[7]+Xj[8],
  {1,2,3}) then return 0 fi;
   if Xj[4]=1 and not member(Xi[4]+Xi[5]+Xj[3]+Xj[5]+Xj[8]+Xj[9],
  {1,2,3}) then return 0 fi;
   if Xj[5]=1 and not member(Xi[5]+Xj[4]+Xj[9]+Xj[10], {1,2,3})
  then return 0 fi;
   1
  end proc:
  T:= Matrix(912,912,Compatible):
```

Thus for $n \geq 2$ $\quad a(n) = \dfrac{u^T T^{n-2} v}{2}$ where u is a column vector with 1 for configurations whose first row could be a top row, 0 otherwise, and similarly v has 1 for configurations whose second row could be a bottom row.

```
> u:= Vector(912,proc(i) local x; x:= Configs[i];
     if x[1]=1 and x[2]+x[6] = 0 then return 0 fi;
     if x[2]=1 and x[1]+x[3]+x[6]+x[7] = 0 then return 0 fi;
     if x[3]=1 and x[2]+x[4]+x[7]+x[8] = 0 then return 0 fi;
     if x[4]=1 and x[3]+x[5]+x[8]+x[9] = 0 then return 0 fi;
     if x[5]=1 and x[4]+x[9]+x[10] = 0 then return 0 fi;
     1
  end proc):
  v:= Vector(912,proc(i) local x; x:= Configs[i];
     if x[6]=1 and x[1]+x[2]+x[7]=0 then return 0 fi;
     if x[7]=1 and x[2]+x[3]+x[6]+x[8]=0 then return 0 fi;
     if x[8]=1 and x[3]+x[4]+x[7]+x[9]=0 then return 0 fi;
     if x[9]=1 and x[4]+x[5]+x[8]+x[10]=0 then return 0 fi;
     if x[10]=1 and x[5]+x[9]=0 then return 0 fi;
     1
   end proc) :
```

To check, here are the first few entries of our sequence (including a_1, which doesn't really fit the pattern, although it will work with the recurrence). For future use, we pre-compute more $T^n v$ than we need.

```
> Tv[0]:= v:
  for n from 1 to 132 do Tv[n]:= T . Tv[n-1] od:
> A:= [12, seq(u^%T . Tv[n],n=0..132)]:
  A[1..20];
```

[12, 494, 9194, 194743, 4254377, 91004329, 1952733110, 41935988643, 900218672974, 19325491197627, 414876662231432, 8906447290570770, 191201120842341933, 4104653067345905449, 88117551274637580431, 1891683137253836217680, 40610128850592429125543, 871806978658994752760570, 18715710335873930220854033, 401783676897552179077879448] **(2)**

To find a recurrence, we look for a linear dependence in the vectors $T^i v$. If $\sum_{i=0}^{n} c_i T^i v = 0$, our recurrence will be $\sum_{i=0}^{n} c_i \, a_{i+n} = 0$

```
> M:= Tv[0]:
  for n from 1 do
    M:= <M|Tv[n]>;
    if LinearAlgebra:-Rank(M) < n+1 then break fi;
  od:
> n;
```

110 **(3)**

Thus there must be a linear recurrence of order 110, i.e. a_{k+110} is a linear combination of $a_k, \ldots, a_{k+109}$.

```
> NM:=LinearAlgebra:-NullSpace(M)[1]:
> NM;
```

$$\begin{bmatrix} -60 \\ 856 \\ -1744 \\ 580 \\ -11559 \\ 12690 \\ 33555 \\ 116269 \\ 79163 \\ -160141 \\ \vdots \end{bmatrix} \tag{4}$$

111 element Vector[column]

Now here is the recurrence formula.

```
> sort(add(NM[i]*a[k+i-1],i=1..111),[seq(a[k+i],i=0..110)])=0;
```

$$-60\, a_{k} + 856\, a_{k+1} - 1744\, a_{k+2} + 580\, a_{k+3} - 11559\, a_{k+4} + 12690\, a_{k+5} + 33555\, a_{k+6} + 116269\, a_{k+7} + 79163\, a_{k+8} - 160141\, a_{k+9} - 2554775\, a_{k+10} - 4257317\, a_{k+11} - 8304527\, a_{k+12} + 9510172\, a_{k+13} + 51868002\, a_{k+14} + 148303995\, a_{k+15} + 228745614\, a_{k+16} + 221065799\, a_{k+17} - 514569184\, a_{k+18} - 1861129043\, a_{k+19} - 4941208137\, a_{k+20} - 7389812306\, a_{k+21} - 8047535244\, a_{k+22} - 2185999847\, a_{k+23} + 28440071472\, a_{k+24} + 93636536075\, a_{k+25} + 236175951848\, a_{k+26} + 461954582082\, a_{k+27} + 794019237745\, a_{k+28} + 1295549706904\, a_{k+29} + 1871390238203\, a_{k+30} + 2259825635674\, a_{k+31} + 1944668702075\, a_{k+32} - 611456662008\, a_{k+33} - 7189868868105\, a_{k+34} - 20866773244914\, a_{k+35} - 43632111285963\, a_{k+36} - 81984655882440\, a_{k+37} - 130412701256303\, a_{k+38} - 199494751280647\, a_{k+39} - 255950296458488\, a_{k+40} - 317259635279467\, a_{k+41} - 323963326279669\, a_{k+42} - 301835023904651\, a_{k+43} - 190865708653347\, a_{k+44} - 6650930016244\, a_{k+45} + 286096836308549\, a_{k+46} + 663659301095746\, a_{k+47} + 1035993556585971\, a_{k+48} + 1519857105478959\, a_{k+49} + 1647382720623854\, a_{k+50} + 1966204561118204\, a_{k+51} + 1705517713004824\, a_{k+52} + 1540166193149230\, a_{k+53} + 833074412333719\, a_{k+54} + 119026455681461\, a_{k+55} - 631761538784627\, a_{k+56} - 1478245466278508\, a_{k+57} - 1598727389561003\, a_{k+58} - 2012497903584757\, a_{k+59} - 1443689140816223\, a_{k+60} - 1481636728203081\, a_{k+61} - 493045406654925\, a_{k+62} - 479981080322206\, a_{k+63} - 36220304582189\, a_{k+64} + 68806466642353\, a_{k+65} + 232142761503906\, a_{k+66} + 81845785738057\, a_{k+67} + 174617559565581\, a_{k+68} + 73371142209400\, a_{k+69} - 249838988495854\, a_{k+70} - 56076320775761\, a_{k+71} \tag{5}$$

$$- 46098148828029\, a_{k+72} + 44755988834179\, a_{k+73} - 10471636917509\, a_{k+74}$$
$$- 20448431604231\, a_{k+75} + 15300794104922\, a_{k+76} - 47631739866152\, a_{k+77}$$
$$+ 46519195647589\, a_{k+78} + 62032906805803\, a_{k+79} - 37462438570926\, a_{k+80}$$
$$- 51892362601875\, a_{k+81} + 8974556479539\, a_{k+82} + 33857629845782\, a_{k+83}$$
$$- 6250593405772\, a_{k+84} - 13344993216692\, a_{k+85} + 4592925731121\, a_{k+86}$$
$$+ 3301298381612\, a_{k+87} - 1543443304990\, a_{k+88} - 963440943120\, a_{k+89}$$
$$+ 691421988338\, a_{k+90} + 87131524306\, a_{k+91} - 192574867282\, a_{k+92}$$
$$+ 46591827116\, a_{k+93} + 17462089147\, a_{k+94} - 12279036687\, a_{k+95}$$
$$+ 1694547297\, a_{k+96} + 678316436\, a_{k+97} - 352926880\, a_{k+98} + 47640203\, a_{k+99}$$
$$+ 7246968\, a_{k+100} - 2562448\, a_{k+101} + 572546\, a_{k+102} - 325799\, a_{k+103} - 37792\, a_{k+104}$$
$$- 31765\, a_{k+105} - 4459\, a_{k+106} - 1345\, a_{k+107} - 106\, a_{k+108} - 13\, a_{k+109} + a_{k+110} = 0$$

We check that it is valid even for $k = 1$.

```
> seq(add(A[k+i]*NM[i],i=1..111),k=0..23);
```

0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 **(6)**