# Maple-assisted derivation of recurrence for A296393

Robert Israel

23 Jun 2020

There are $2^{10} = 1024$ configurations for a $2 \times 5$ sub-array, but not all can arise.

We encode these configurations as lists in the order $\begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ x_6 & x_7 & x_8 & x_9 & x_{10} \end{bmatrix}$. Each of $x_2, x_3, x_4, x_7, x_8, x_9$ is horizontally, vertically or diagonally adjacent to 4 other entries, and those (plus the entry itself) can't all be 1's.

```
> goodconfig:= proc(x)
     x[2]+x[1]+x[3]+x[6]+x[7]<=4
     and x[3]+x[2]+x[4]+x[7]+x[8]<=4
     and x[4]+x[3]+x[5]+x[8]+x[9]<=4
     and x[7]+x[2]+x[3]+x[6]+x[8]<=4
     and x[8]+x[3]+x[4]+x[7]+x[9]<=4
     and x[9]+x[4]+x[5]+x[8]+x[10]<=4
  end proc:
  Configs:= select(goodconfig, [seq(convert(2^10+i,base,2)[1..10],
  i=0..2^10-1)]):
  nops(Configs);
```

$$912 \tag{1}$$

There are 912 allowed configurations.

Consider the $912 \times 912$ transition matrix $T$ with entries $T_{ij} = 1$ if the first two rows of a $3 \times 5$ sub-array could be in configuration $i$ while the last two rows are in configuration $j$, and 0 otherwise. The following code computes it.

```
> Compatible:= proc(i,j) local Xi,Xj,k;
   Xi:= Configs[i]; Xj:= Configs[j];
   if Xi[6..10] <> Xj[1..5] then return 0 fi;
   if Xj[1]=1 and not member(Xi[1]+Xi[2]+Xj[2]+Xj[6], {1,3}) then
  return 0 fi;
   if Xj[2]=1 and not member(Xi[2]+Xi[3]+Xj[1]+Xj[3]+Xj[6]+Xj[7],
  {1,3}) then return 0 fi;
   if Xj[3]=1 and not member(Xi[3]+Xi[4]+Xj[2]+Xj[4]+Xj[7]+Xj[8],
  {1,3}) then return 0 fi;
   if Xj[4]=1 and not member(Xi[4]+Xi[5]+Xj[3]+Xj[5]+Xj[8]+Xj[9],
  {1,3}) then return 0 fi;
   if Xj[5]=1 and not member(Xi[5]+Xj[4]+Xj[9]+Xj[10], {1,3}) then
  return 0 fi;
   1
  end proc:
  T:= Matrix(912,912,Compatible):
```

Thus for $n \geq 2$ $a(n) = \dfrac{u^T T^{n-2} v}{2}$ where $u$ is a column vector with 1 for configurations whose first row could be a top row, 0 otherwise, and similarly $v$ has 1 for configurations whose second row could be a bottom row.

```
> u:= Vector(912,proc(i) local x; x:= Configs[i];
     if x[1]=1 and x[2]+x[6] <> 1 then return 0 fi;
     if x[2]=1 and not member(x[1]+x[3]+x[6]+x[7], {1,3}) then
  return 0 fi;
     if x[3]=1 and not member(x[2]+x[4]+x[7]+x[8], {1,3}) then
  return 0 fi;
     if x[4]=1 and not member(x[3]+x[5]+x[8]+x[9], {1,3}) then
  return 0 fi;
     if x[5]=1 and not member(x[4]+x[9]+x[10], {1,3}) then return 0
  fi;
     1
  end proc):
  v:= Vector(912,proc(i) local x; x:= Configs[i];
     if x[6]=1 and not member(x[1]+x[2]+x[7],{1,3}) then return 0
  fi;
     if x[7]=1 and not member(x[2]+x[3]+x[6]+x[8],{1,3}) then
  return 0 fi;
     if x[8]=1 and not member(x[3]+x[4]+x[7]+x[9],{1,3}) then
  return 0 fi;
     if x[9]=1 and not member(x[4]+x[5]+x[8]+x[10],{1,3}) then
  return 0 fi;
     if x[10]=1 and x[5]+x[9]<> 1 then return 0 fi;
     1
   end proc) :
```

To check, here are the first few entries of our sequence (including $a_1$, which doesn't really fit the pattern, although it will work with the recurrence). For future use, we pre-compute more $T^n v$ than we need.

```
> Tv[0]:= v:
  for n from 1 to 132 do Tv[n]:= T . Tv[n-1] od:
> A:= [6, seq(u^%T . Tv[n],n=0..132)]:
  A[1..20];
```

[6, 63, 387, 2969, 21769, 159732, 1181174, 8690359, 64100854, 472391582, 3481701205, 25663390294, 189149912070, 1394165032202, 10275815563148, 75739024915619, 558242966775489, 4114589197600076, 30327032266275765, 223528681117250009] **(2)**

To find a recurrence, we look for a linear dependence in the vectors $T^i v$. If $\sum_{i=0}^{n} c_i T^i v = 0$, our recurrence will be $\sum_{i=0}^{n} c_i\, a_{i+n} = 0$

```
> M:= Tv[0]:
  for n from 1 do
    M:= <M|Tv[n]>;
    if LinearAlgebra:-Rank(M) < n+1 then break fi;
  od:
> n;
```

125 **(3)**

Thus there must be a linear recurrence of order 125, i.e. $a_{k+125}$ is a linear combination of $a_k, \ldots, a_{k+124}$.

```
> NM:=LinearAlgebra:-NullSpace(M)[1]:
> NM;
```

$$\begin{bmatrix} 1 \\ 13 \\ -75 \\ 211 \\ -952 \\ 3607 \\ -11899 \\ 37139 \\ -77533 \\ 227127 \\ \vdots \end{bmatrix} \quad (4)$$

126 element Vector[column]

Now here is the recurrence formula.

```
> sort(add(NM[i]*a[k+i-1],i=1..126),[seq(a[k+i],i=0..125)])=0;
```

$$a_k + 13\, a_{k+1} - 75\, a_{k+2} + 211\, a_{k+3} - 952\, a_{k+4} + 3607\, a_{k+5} - 11899\, a_{k+6} + 37139\, a_{k+7} - 77533\, a_{k+8} + 227127\, a_{k+9} - 567844\, a_{k+10} + 631217\, a_{k+11} - 1929222\, a_{k+12} + 3687457\, a_{k+13} - 630641\, a_{k+14} + 2476903\, a_{k+15} - 4649609\, a_{k+16} - 20182867\, a_{k+17} + 32428305\, a_{k+18} + 60624119\, a_{k+19} + 87680796\, a_{k+20} - 6077658\, a_{k+21} - 89255695\, a_{k+22} - 978559995\, a_{k+23} + 457960985\, a_{k+24} - 891814277\, a_{k+25} + 1590343633\, a_{k+26} + 1431180031\, a_{k+27} - 5714536093\, a_{k+28} - 5528911999\, a_{k+29} - 20431516979\, a_{k+30} - 10510060646\, a_{k+31} + 1850187776\, a_{k+32} + 47937518084\, a_{k+33} + 82919115596\, a_{k+34} + 160103639097\, a_{k+35} + 164422661069\, a_{k+36} + 201841378402\, a_{k+37} + 171844743778\, a_{k+38} + 46885338658\, a_{k+39} - 68222315027\, a_{k+40} - 524026642801\, a_{k+41} - 717166675018\, a_{k+42} - 1389539772791\, a_{k+43} - 1037511235951\, a_{k+44} - 1186221021965\, a_{k+45} + 371383054554\, a_{k+46} + 878091852190\, a_{k+47} + 2424737607675\, a_{k+48} + 1409056986740\, a_{k+49} + 897934434295\, a_{k+50} - 3248589008156\, a_{k+51} - 4802732806331\, a_{k+52} - 8510819698703\, a_{k+53} - 7311711493597\, a_{k+54} - 5891307676340\, a_{k+55} - 1687099916159\, a_{k+56} + 2467556957286\, a_{k+57} + 4458566790574\, a_{k+58} + 5371398004695\, a_{k+59} + 1768020926908\, a_{k+60} - 447096752663\, a_{k+61} - 5305712432995\, a_{k+62} - 6115620347627\, a_{k+63} - 6911890534168\, a_{k+64} - 4821064615061\, a_{k+65} - 2794761636109\, a_{k+66} - 587557346812\, a_{k+67} + 893469383568\, a_{k+68} + 1111603699677\, a_{k+69} + 1616862353455\, a_{k+70} + 1043280543415\, a_{k+71} + 1326609919732\, a_{k+72} + 1129216580786\, a_{k+73} + 1363490591266\, a_{k+74} \quad (5)$$

$$+ 1116695860020\, a_{k+75} + 977398224998\, a_{k+76} + 628498276069\, a_{k+77}$$
$$+ 218217221933\, a_{k+78} + 55178404635\, a_{k+79} - 208368890821\, a_{k+80}$$
$$- 195851412004\, a_{k+81} - 243186392350\, a_{k+82} - 195127888385\, a_{k+83}$$
$$- 165491916397\, a_{k+84} - 119229963803\, a_{k+85} - 66848143635\, a_{k+86}$$
$$- 48347546976\, a_{k+87} - 7627237475\, a_{k+88} + 7871124360\, a_{k+89} + 6468530288\, a_{k+90}$$
$$+ 10627036764\, a_{k+91} + 12399903998\, a_{k+92} + 4721946713\, a_{k+93} + 2136727497\, a_{k+94}$$
$$+ 2233335183\, a_{k+95} + 48426922\, a_{k+96} - 2037325637\, a_{k+97} - 838475623\, a_{k+98}$$
$$- 204601423\, a_{k+99} - 530454296\, a_{k+100} + 7559429\, a_{k+101} + 264062098\, a_{k+102}$$
$$+ 84146824\, a_{k+103} + 3629098\, a_{k+104} + 49724318\, a_{k+105} + 17424184\, a_{k+106}$$
$$- 19950209\, a_{k+107} - 5074710\, a_{k+108} + 2255133\, a_{k+109} - 1717230\, a_{k+110}$$
$$- 553186\, a_{k+111} + 190763\, a_{k+112} - 252596\, a_{k+113} - 1787\, a_{k+114} + 36111\, a_{k+115}$$
$$+ 6261\, a_{k+116} + 11759\, a_{k+117} + 4770\, a_{k+118} - 304\, a_{k+119} - 309\, a_{k+120} - 59\, a_{k+121}$$
$$- 98\, a_{k+122} - 32\, a_{k+123} - a_{k+124} + a_{k+125} = 0$$

We check that it is valid even for $k = 1$.

```
> seq(add(A[k+i]*NM[i],i=1..126),k=0..8);
```

0, 0, 0, 0, 0, 0, 0, 0, 0 **(6)**