Maple-assisted proof of formula for A295547

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There are $2^8 = 256$ possible configurations for a 2×4 sub-array. Consider the 256×256 transition matrix T such that $T_{ij} = 1$ if the bottom two rows of a 3×4 sub-array could be in configuration i while the top two rows are in configuration j (i.e. the middle row is compatible with both i and j, and each 1 in that row is horizontally or vertically adjacent to 1,3 or 4 1's), and 0 otherwise. The following Maple code computes it. I'm encoding a configuration

$$\begin{bmatrix} b_1 & b_2 & b_3 & b_4 \\ b_5 & b_6 & b_7 & b_8 \end{bmatrix}$$

as $b+1$ where $b_1 b_2 b_3 b_4 b_5 b_6 b_7 b_8$ is the binary representation of b. The $+1$ is needed because matrix indices start at 1 rather than 0.

```
> q:= proc(a,b) local r,s,t,M,i;
    s:= floor((a-1)/16);
    if s <> (b-1) mod 16 then return 0 fi;
    s:= convert(s+16,base,2);
    r:= convert(16+floor((b-1)/16),base,2);
    t:= convert(16+ ((a-1) mod 16),base,2);
    M:= Vector(4);
    for i from 1 to 3 do if s[i] = 1 and s[i+1] = 1 then M[i]:= M
 [i]+1; M[i+1]:= 1 fi od;
    for i from 1 to 4 do if s[i]=1 then
      M[i]:= M[i]+r[i]+t[i];
      if M[i] =0 or M[i]=2 then return 0 fi;
    fi od;
    1
  end proc:
  T:= Matrix(256,256, q);
```

$$T := \begin{bmatrix} \textit{256 x 256 Matrix} \\ \textit{Data Type: anything} \\ \textit{Storage: rectangular} \\ \textit{Order: Fortran\_order} \end{bmatrix} \quad \textbf{(1)}$$

Thus $a(n) = u\,T^n\,v$ where u and v are row and column vectors respectively with $u_i = 1$ for i corresponding to configurations with bottom row $(0, 0, 0, 0)$, 0 otherwise, and $v_i = 1$ for i corresponding to configurations with top row $(0, 0, 0, 0)$, 0 otherwise. The following Maple code produces these vectors.

```
> u:= Vector[row](256):
  v:= Vector(256):
  for i from 0 to 15 do u[16*i+1]:= 1; v[i+1]:= 1;
  od:
```

To check, here are the first few entries of our sequence.

```
> seq(u . T^n . v, n = 1 .. 10);
```

$$4, 24, 116, 598, 3035, 15352, 78434, 399324, 2032606, 10348672 \quad (2)$$

Now here is the minimal polynomial P of T, as computed by Maple.

```
> P:= unapply(LinearAlgebra:-MinimalPolynomial(T, t), t);
```

$$P := t \rightarrow -t + t^{82} - 7 t^{81} + 7 t^{80} + 11 t^{79} + 4 t^{78} + 89 t^{77} - 111 t^{76} - 247 t^{75} - 803 t^{74} - 151 t^{73} + 6320 t^{72} + 5635 t^{71} - 47068 t^{70} - 40186 t^{69} + 235934 t^{68} + 143812 t^{67} - 534069 t^{66} - 182815 t^{65} + 345119 t^{64} - 226999 t^{63} + 447016 t^{62} + 906019 t^{61} + 1029283 t^{60} + 703135 t^{59} - 3984038 t^{58} - 5372312 t^{57} - 3757126 t^{56} + 92899 t^{55} + 4711665 t^{54} + 1648318 t^{53} + 8451237 t^{52} + 9703424 t^{51} - 5867969 t^{50} - 3136527 t^{49} + 3614506 t^{48} + 8113673 t^{47} + 6314777 t^{46} - 4266717 t^{45} - 6171438 t^{44} - 29491646 t^{43} - 2614062 t^{42} - 5543573 t^{41} + 12047674 t^{40} + 18036657 t^{39} + 15291458 t^{38} + 36461541 t^{37} - 20327725 t^{36} + 22164840 t^{35} - 50041770 t^{34} - 10027137 t^{33} - 44382281 t^{32} - 14766373 t^{31} - 17673316 t^{30} + 1851348 t^{29} + 5467781 t^{28} + 19874308 t^{27} + 9645362 t^{26} + 22815895 t^{25} + 4046955 t^{24} + 13403751 t^{23} - 435715 t^{22} + 4213227 t^{21} - 1442145 t^{20} + 5322 t^{19} - 675097 t^{18} - 617091 t^{17} - 85131 t^{16} - 254478 t^{15} + 34007 t^{14} - 46131 t^{13} + 12845 t^{12} + 992 t^{11} + 491 t^{10} + 2725 t^{9} - 225 t^{8} + 394 t^{7} + 50 t^{6} - 30 t^{5} + 5 t^{4} + 14 t^{3} \quad (3)$$

This turns out to have degree 82. Thus we will have $0 = u\,P(T)\;T^n\,v = \sum_{i=0}^{82} p_i\,a(i+n)$ where p_i is the coefficient of t^i in $P(t)$. That corresponds to a homogeneous linear recurrence of order 82, which would hold true for any u and v. It seems that with our particular u and v we have a recurrence of order only 45, corresponding to a factor of P.

```
> factor(P(t));
```

$$t\,(t^{36} - t^{34} - 3 t^{33} + t^{32} + 43 t^{31} - 39 t^{30} - 137 t^{29} + 42 t^{28} + 205 t^{27} + 330 t^{26} - 132 t^{25} - 941 t^{24} - 723 t^{23} + t^{22} + 960 t^{21} + 1183 t^{20} - 82 t^{19} - 32 t^{18} - 739 t^{17} - 883 t^{16} + 206 t^{15} - 764 t^{14} + 684 t^{13} - 276 t^{12} + 771 t^{11} + 91 t^{10} + 281 t^{9} + 61 t^{8} - 52 t^{7} - 9 t^{6} - 43 t^{5} - t^{4} - 8 t^{3} + t^{2} + t - 1)\,(t^{45} - 7 t^{44} + 8 t^{43} + 7 t^{42} - 10 t^{41} + 84 t^{40} + 232 t^{39} - 680 t^{38} - 1299 t^{37} + 1669 t^{36} + 475 t^{35} - 3475 t^{34} + 5871 t^{33} + 8620 t^{32} - 4957 t^{31} + 3601 t^{30} + 7232 t^{29} - 1897 t^{28} - 10512 t^{27} - 544 t^{26} - 795 t^{25} - 21086 t^{24} - 6396 t^{23} - 19576 t^{22} - 6829 t^{21} - 16929 t^{20} + 15374 t^{19} - 664 t^{18} + 12805 t^{17} + 19325 t^{16} + 17150 t^{15} + 12843 t^{14} + 15082 t^{13} + 5335 t^{12} + 8181 t^{11} + 58 t^{10} + 3152 t^{9} - 1066 t^{8} + 493 t^{7} - 294 t^{6} - 37 t^{5} - 15 t^{4} - 24 t^{3} - 12 t^{2} + t + 1) \quad (4)$$

```
> Q:= unapply(t^45-7*t^44+8*t^43+7*t^42-10*t^41+84*t^40+232*t^39
-680*t^38-1299*t^37+1669*t^36+475*t^35-3475*t^34+5871*t^33+8620*
t^32-4957*t^31+3601*t^30+7232*t^29-1897*t^28-10512*t^27-544*t^26
-795*t^25-21086*t^24-6396*t^23-19576*t^22-6829*t^21-16929*
t^20+15374*t^19-664*t^18+12805*t^17+19325*t^16+17150*t^15+12843*
t^14+15082*t^13+5335*t^12+8181*t^11+58*t^10+3152*t^9-1066*
t^8+493*t^7-294*t^6-37*t^5-15*t^4-24*t^3-12*t^2+t+1, t);
```

(5)

$$Q := t \to 1 + t - 12\,t^2 + t^{45} - 7\,t^{44} + 8\,t^{43} + 7\,t^{42} - 10\,t^{41} + 84\,t^{40} + 232\,t^{39} - 680\,t^{38} - 1299\,t^{37} + 1669\,t^{36} + 475\,t^{35} - 3475\,t^{34} + 5871\,t^{33} + 8620\,t^{32} - 4957\,t^{31} + 3601\,t^{30} + 7232\,t^{29} - 1897\,t^{28} - 10512\,t^{27} - 544\,t^{26} - 795\,t^{25} - 21086\,t^{24} - 6396\,t^{23} - 19576\,t^{22} - 6829\,t^{21} - 16929\,t^{20} + 15374\,t^{19} - 664\,t^{18} + 12805\,t^{17} + 19325\,t^{16} + 17150\,t^{15} + 12843\,t^{14} + 15082\,t^{13} + 5335\,t^{12} + 8181\,t^{11} + 58\,t^{10} + 3152\,t^9 - 1066\,t^8 + 493\,t^7 - 294\,t^6 - 37\,t^5 - 15\,t^4 - 24\,t^3 \quad \textbf{(5)}$$

The complementary factor $R(t) = \dfrac{P(t)}{Q(t)}$ has degree 37.

```
> R:= unapply(normal(P(t)/Q(t)), t);
```

$$R := t \to \left(-1 + t + t^2 + t^{36} - t^{34} - 3\,t^{33} + t^{32} + 43\,t^{31} - 39\,t^{30} - 137\,t^{29} + 42\,t^{28} + 205\,t^{27} + 330\,t^{26} - 132\,t^{25} - 941\,t^{24} - 723\,t^{23} + t^{22} + 960\,t^{21} + 1183\,t^{20} - 82\,t^{19} - 32\,t^{18} - 739\,t^{17} - 883\,t^{16} + 206\,t^{15} - 764\,t^{14} + 684\,t^{13} - 276\,t^{12} + 771\,t^{11} + 91\,t^{10} + 281\,t^9 + 61\,t^8 - 52\,t^7 - 9\,t^6 - 43\,t^5 - t^4 - 8\,t^3\right) t \quad \textbf{(6)}$$

Now we want to show that $b(n) = u\,Q(T)\,T^n\,v = 0$ for all n. This will certainly satisfy the order-37 recurrence

$$\sum_{i=0}^{37} r_i\,b(i+n) = \sum_{i=0}^{37} r_i u\,Q(T)\,T^{n+i} v = u\,Q(T)\,R(T)\,T^n\,v = u\,P(T)\,T^n\,v = 0$$

where r_i are the coefficients of $R(t)$. To show all $b(n) = 0$ it suffices to show $b(0) = ... = b(36) = 0$. This would take some time to do naively, so it's worthwhile to do some pre-calculation.

```
> w:= u . Q(T):
  V[0]:= v:
  for i from 1 to 36 do V[i]:= T . V[i-1] od:
  seq(w . V[i],i=0..36);
```

0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 **(7)**