Proof of a conjecture stated in A293129

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Let $(a(n))_{n\in\mathbb{N}}$ be the sequence whose logarithmic generating function

$$L(x)=\sum_{n\in\mathbb{N}} a(n)\frac{x^{2n-1}}{2n-1}, \tag{1}$$

is given by

$$L(x)=\sum_{m=-\infty}^{\infty}\frac{(x-x^{2m-1})^{2m-1}}{2m-1}.$$

The statement of the following theorem was conjectured by Hanna in A293129.

Theorem 1. *Let $n\in\mathbb{N}$. Then $a(2^n+1)=1$.*

Proof. According to (1), $a(2^n+1)=(2^{n+1}+1)[x^{2^{n+1}+1}]L(x)$. By a comment in A293129,

$$L(x)=-\log(1-x)-\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}\frac{(x-x^{2m})^{2m}}{2m}.$$

Since $-\log(1-x)=\sum_{m=1}^{\infty}\frac{x^m}{m}$, we have $[x^{2^{n+1}+1}](-\log(1-x))=\frac{1}{2^{n+1}+1}$. Let $0\neq m\in\mathbb{Z}$ and distinguish between two cases.

1. $m\geq 1$. We have

$$(x-x^{2m})^{2m}=x^{2m}(1-x^{2m-1})^{2m}=\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}x^{2m+k(2m-1)}.$$

   Assume that for some $0\leq k\leq 2m$, we have

$$2^{n+1}+1=2m+k(2m-1). \tag{2}$$

   Reducing (2) modulo $2m-1$ yields $2^{n+1}+1\equiv 1 \pmod{2m-1}$. Thus, $(2m-1)\mid 2^{n+1}$. But $2m-1$ is odd while 2^{n+1} is a power of 2.

Thus, necessarily, $m = 1$. Thus, (2) has the form $2^{n+1} + 1 = 2 + k$, i.e., $k = 2^{n+1} - 1$. We have $2^{n+1} - 1 \geq 3 > 2 = 2m$, contradicting $k \leq 2m$. We have therefore proved that, for every $m \geq 1$, we have $[x^{2^{n+1}+1}](x - x^{2m})^{2m} = 0$.

2. $m \leq -1$. Set $u = -m$. Then

$$(x - x^{2m})^{2m} = x^{4u^2}(1 - x^{2u+1})^{-2u} = \sum_{k=0}^{\infty} \binom{2u+k-1}{k} x^{4u^2+k(2u+1)}.$$

Assume that for some $k \in \mathbb{N}_0$ we have

$$\begin{aligned} 2^{n+1} + 1 &= 4u^2 + k(2u+1) \\ &= (2u-1)(2u+1) + 1 + k(2u+1). \end{aligned} \tag{3}$$

Reducing (3) modulo $2u + 1$ yields $2^{n+1} + 1 \equiv 1 \pmod{2u+1}$. Thus, $(2u + 1) \mid 2^{n+1}$. But $2u + 1 \geq 3$ is odd, while 2^{n+1} is a power of 2. Thus, (3) can not hold. We have therefore proved that, for every $m \leq -1$, we have $[x^{2^{n+1}+1}](x - x^{2m})^{2m} = 0$.

It follows that

$$\begin{aligned} [x^{2^{n+1}+1}]L(x) &= [x^{2^{n+1}+1}](-\log(1-x)) - [x^{2^{n+1}+1}] \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{(x - x^{2m})^{2m}}{2m} \\ &= \frac{1}{2^{n+1}+1} + 0 = \frac{1}{2^{n+1}+1}, \end{aligned}$$

proving the assertion. □

References

[1] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, OEIS Foundation Inc., `https://oeis.org`.