NOTES ON $(\sigma + \varphi)/2$

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Define $f(n) = \frac{1}{2}(\sigma(n) + \varphi(n))$ for $n \in \mathbb{N}$.

Lemma 1. *$f(n) \geq n$ for all $n \in \mathbb{N}$, with equality if and only if n is a prime number or 1.*

Proof. Note that $f(1) = 1$. For $n > 1$ we may write $n = p^k m$ for some prime power p^k and $m \in \mathbb{N}$ with $p \nmid m$. Then a computation shows that

$$f(n) = p^k f(m) + p^{k-1}\frac{\sigma(m) - \varphi(m)}{2} + \frac{p^{k-1} - 1}{2(p-1)}\sigma(m) \geq p^k f(m),$$

with equality if and only if n is prime. The lemma follows by induction. $\square$

Lemma 2. *For $n \in \mathbb{N}$, $f(n) \in \mathbb{N}$ if and only if $n \notin \{m^2, 2m^2 : m > 1\}$.*

Lemma 3. *If $n > 1$ and $f(n) \in 1 + 2\mathbb{Z}$ then $n \in \{pm^2, 2pm^2 : p$ odd prime, $m \in \mathbb{N}\}$.*

Proof. Let $n > 1$ with $f(n)$ an odd integer. Since $f(2) = 2$, we have $n > 2$, so that $\varphi(n)$ is even. Since $f(n)$ is odd, we have

$$\sigma(n) + \varphi(n) \equiv 2 \pmod 4.$$

If $4 \mid \varphi(n)$ then $\sigma(n) \equiv 2 \pmod 4$, which can only happen if n has exactly one odd prime factor of odd multiplicity, which implies that n is of the required form. If $4 \nmid \varphi(n)$ then $\varphi(n) \equiv 2 \pmod 4$, and this can happen only if $n \in \{4, p^e, 2p^e : p \text{ odd prime}, \ e \in \mathbb{N}\}$. By Lemma 2, we may assume that $n \neq 4$ and e is odd, so again n is of the required form. $\square$

Proposition 4. *For $x > 1$,*

$$\#\{n \leq x : n \text{ composite and } f(n) \text{ prime}\} \ll \frac{x}{\log^2 x}.$$

Proof. Suppose that n is composite and $f(n)$ is prime. Then $f(n) > n \geq 4$, so $f(n)$ must be an odd prime. By Lemma 3, it follows that $n = pm$, where p is an odd prime and $m > 1$ is a square or twice a square. If $p \mid m$ then either n or $n/2$ is squarefull, and the number of such $n \leq x$ is $O(\sqrt{x})$. Hence we may assume that $p \nmid m$. In that case, writing $p = 2t + 1$ for $t \in \mathbb{N}$, we have $f(n) = at + b$, where $a = \sigma(m) + \varphi(m)$ and $b = \sigma(m)$. If $(a, b) > 1$ then $at + b$ is never prime, so we may assume that $(a, b) = 1$.

For a prime q, let $\omega(q)$ denote the number of solutions $t \pmod q$ of the congruence

$$(2t + 1)(at + b) \equiv 0 \pmod q.$$

Then we compute that

$$\omega(q) = \begin{cases} 0 & \text{if } m = q = 2, \\ 1 & \text{if } m > q = 2, \\ 1 & \text{if } q > 2 \text{ and } \sigma(m)^2 \equiv \varphi(m)^2 \pmod q, \\ 2 & \text{otherwise.} \end{cases}$$

Applying [1, Ch. II, Satz 4.2], we find that

$$\#\{p \le x/m : f(pm) \text{ prime}\} \le \#\{t \le x/2m : 2t+1 \text{ and } at+b \text{ prime}\}$$
$$\ll \frac{x/2m}{\log^2(x/2m)} \prod_{q \mid (\sigma(m)^2-\varphi(m)^2)} \frac{q}{q-1} \ll \frac{x \log\log(3m)}{m \log^2(x/2m)}.$$

Summing this estimate over all squares and twice squares $m \in [2, x/3]$ gives $O(x/\log^2 x)$, as required. □

References

1. Karl Prachar, *Primzahlverteilung*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 91, Springer-Verlag, Berlin-New York, 1978, Reprint of the 1957 original. MR 516660