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THE AT MOST UNICYCLIC RANDOM GRAPH PROCESS

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Abstract

A random graph process is defined in the following way. Let $\mathcal{U}$ denote a set of unlabeled graphs of order *n*. A graph G_0 not equal to the empty graph is called an *initial graph*, if the deletion of any edge of G_0 produces a graph not in $\mathcal{U}$. If $\mathcal{U}$ contains the empty graph then the empty graph is an initial graph in $\mathcal{U}$.

Starting at an initial graph G_0 in $\mathcal{U}$, *randomly*, that is with equally likely probability, add an edge $\{u, v\}$ to start a random walk (G_i) such that at each step $G_{i+1} = G_i \cup \{u, v\}$ is in $\mathcal{U}$ for all $i \geq 0$.

Let N be the number of edges such that $G_{i+1} = G_i \cup \{u, v\}$ is in $\mathcal{U}$. Such edges are called *admissible edges*. For N $\neq$ 0, the probability that edge $\{u, v\}$ is selected is 1/N. If N = 0, then the graph G_i is called a *terminal graph.*

The probability of going from a non-terminal graph G_i to G_{i+1} is called the *transition probability* from G_i to G_{i+1}. This probability is the number of edges in G_i that produce G_{i+1} divided by the number of admissible edges in G_i.

The *transition digraph* for this random process is the union of all random walks described above. Namely, the node set for the transition digraph is $\mathcal{U}$ and the arc set consists of the arcs (G_i, G_{i+1}) weighted with their corresponding transition probabilities.

In this paper $\mathcal{U}$ is defined as the set of unlabeled graphs of order *n* having at most one cycle to obtain the *At Most Unicyclic Random Graph Process.*The properties of this random process and in particular the properties and structure of its associated transition digraph are studied.

1. Introduction

Let $\mathcal{U}$ denote the set of unlabeled graphs of order n having at most one cycle (such graphs are called *at most unicyclic*). A graph G_0 in $\mathcal{U}$ is called an *initial graph* in $\mathcal{U}$, if the deletion of any edge of G_0 produces a graph not in $\mathcal{U}$. If the *empty graph*, the graph with no edges and n vertices, is in $\mathcal{U}$, then the empty graph is an initial graph. In the at most unicyclic case, the empty graph is the unique initial graph. This is in contrast to the unicyclic case where there are multiple initial graphs (see **[1]**). Next, starting at the unique initial graph $G_0 = nK_1$ in $\mathcal{U}$, randomly, that is, with equally likely probability, add an edge $\{u, v\}$ to start a random walk (G_i) such that at each step

$$G_{i+1} = G_i \cup \{u, v\} \text{ is in } \mathcal{U} \text{ for all } i \geq 0.$$

The probability that edge $\{u, v\}$ in G_i is selected is 1/N, where N $\neq$ 0 is the number of edges such that $G_{i+1} = G_i \cup \{u, v\}$ is in $\mathcal{U}$. Such edges are called *admissible edges*. If N = 0, then the graph G_i is called a *terminal graph.* Note that the connected unicyclic graphs are the terminal graphs.

The *transition digraph* TD(n) for this random process is the union of all random walks described above. Namely, the node set for the transition digraph is $\mathcal{U}$ and the arc set consists of the arcs (G_i, G_{i+1}) weighted with their corresponding transition probabilities.

For example, the transition probability for going from the graph G_i consisting of the 3-cycle union $n - 3$ isolated vertices to the graph G_{i+1} consisting of a 3-cycle union an edge union $n - 5$ isolated vertices for $n \geq 5$ is obtained as follows.

There are C(n – 3, 2) ways of introducing an isolated edge to G_i. The total number of ways of introducing an edge to G_i is 3(n – 3) + C(n – 3, 2) = N, the number of admissible edges. Thus, the transition probability of going from G_i to G_{i+1} is C(n – 3, 2) / N, which simplifies to (n – 4) / (n + 2).

In this paper the enumerations of the order, size, and terminal graphs of TD(n) are studied. The planarity of TD(n) is determined. Walks in the underlying digraph and underlying graph of TD(n) are investigated. Results concerning the traceability of the underlying graph of TD(n) are obtained. Also studied are the probability distributions of various classes of graphs in $\mathcal{U}$ relative to the random graph process of the title. For example, the probability distribution of the terminal graphs is determined. As a result of these studies a number of open problems are formulated.

2. The At Most Unicyclic Random Graph Process

In this section some properties of the at most unicyclic random graph process will be determined.

(2.1) Initial graphs

As already noted in the introduction this random process has only one initial graph, namely, the empty graph nK_1. This is clear since the deletion of any edge in a nonempty member of $\mathcal{UC}$ results in a member of $\mathcal{UC}$.

(2.2) Terminal graphs and their number

Also noted in the introduction, the terminal graphs in this process are the connected unicyclic graphs of order *n*.

Theorem 2.2.1. The terminal graphs for the at most unicyclic random graph process are the connected unicyclic graphs.

Proof. This is seen by noting that after *n* steps (*n* edge insertions) starting from the initial empty graph, the graphs (nodes in TD(*n*)) on level *n* have *n* vertices and *n* edges. Due to the unicyclic condition, these graphs must be connected and unicyclic. The further insertion of any edge will result in the addition of one or more cycles. Thus, the graphs on level *n* are terminal graphs. ■

These graphs have been enumerated and the sequence of numbers of graphs for each order can be found as sequence A001429 in **[2]**. See **Table 2.2.1** below for values for $3 \leq n \leq 14$.

Table 2.2.1 The number *t* of unlabeled connected unicyclic graphs of order *n*, $3 \leq n \leq 14$

n	3	4	5	6	7	8	9	10	11	12	13	14
t	1	2	5	13	33	89	240	657	1806	5026	13999	39260

(2.3) Order of the transition digraph

The order of the transition digraph TD(*n*) is by definition the number of graphs in $\mathcal{UC}$, that is, the number *v* of unlabeled graphs of order *n* having at most one cycle (see **Table 2.3.3**). The set $\mathcal{UC}$ can be partitioned into the set of unicyclic graphs of order *n* union the set of forests of order *n*. Since both of these sets have been enumerated, the number *v* is obtained by simple addition of their respective numbers for a given *n*. For the number *a* of unlabeled unicyclic graphs of order *n*, see sequence A236570 in **[2]**. **Table 2.3.1** below lists these values for $3 \leq n \leq 14$. For the number b of unlabeled forests of order n, see sequence A005195 in **[2]**. **Table 2.3.2** below lists these values for $3 \leq n \leq 14$. **Table 2.3.3** is the result of $v = a + b$.

Table 2.3.1 The number a of unlabeled unicyclic graphs of order n, $3 \leq n \leq 14$

n	3	4	5	6	7	8	9	10	11	12	13	14
a	1	3	9	25	68	185	504	1379	3788	10480	29094	81193

Table 2.3.2 The number b of unlabeled forests of order n, $3 \leq n \leq 14$

n	3	4	5	6	7	8	9	10	11	12	13	14
b	3	6	10	20	37	76	153	329	710	1601	3658	8599

Table 2.3.3 The number v of graphs of order n having at most one cycle, $3 \leq n \leq 14$

n	3	4	5	6	7	8	9	10	11	12	13	14
v	4	9	19	45	105	261	657	1708	4498	12081	32752	89792

The sequence v for the number of at most unicyclic graphs of order n was submitted to *The On-Line Encyclopedia of Integer Sequences* **[2]**. The response was that no matches were found.

(2.4) Size of the transition digraph

The size of TD(n) is not immediately apparent. However, the size is known for small n (1, 2, and 3) see **Figure 2.4.1**, $n = 4$ see **Figure 2.5.2,** and for $n = 5$ see the transition matrix shown in **Section 2.7**.

Note that at each terminal graph there is placed a loop with probability one. This indicates that a random walk remains at that terminal graph for all steps greater than n. The loop is counted as an arc.

Table 2.4.1 The number E of arcs in TD(n) for n, $1 \leq n \leq 5$

n	1	2	3	4	5
E	1	2	4	13	40

No closed form for the size of TD(n) is known. In a comprehensive study of random graph processes involving graphs with bounded degree, algorithms are given for obtaining the order and size of the transition digraphs for these random graph processes **[6]**. Possibly small modifications of these algorithms could yield solutions to the following problem.

Problem 2.1 What is the size of TD(n) for $n \geq 6$?

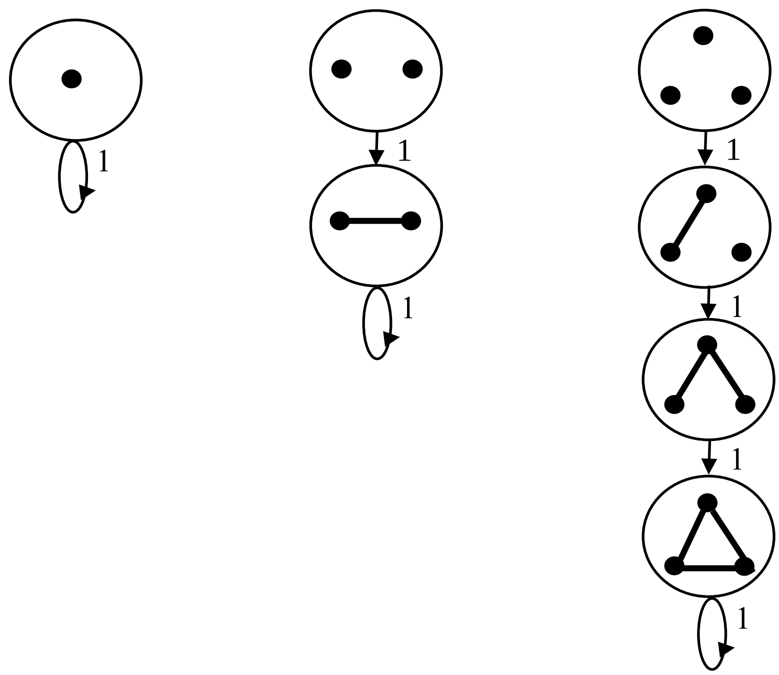

Figure 2.4.1 TD(n) for n = 1, 2, and 3

(2.5) Planarity

Theorem 2.5.1 TD(n) is nonplanar for $n \geq 5$ and planar for $1 \leq n \leq 4$.

Proof. The digraph shown in **Figure 2.5.1** is a subgraph of TD(5) and is homeomorphic to $K_{3,3}$. Thus, TD(5) is nonplanar. Since TD(n) is isomorphic to a subgraph of TD(n + 1) for all n, it follows that TD(n) is nonplanar for $n \geq 5$.

One has TD(4) is planar (see **Figure 2.5.2**) and TD(n) for $1 \leq n \leq 3$ is planar (see **Figure 2.4.1**). Thus, TD(n) for $1 \leq n \leq 4$ is planar. ■

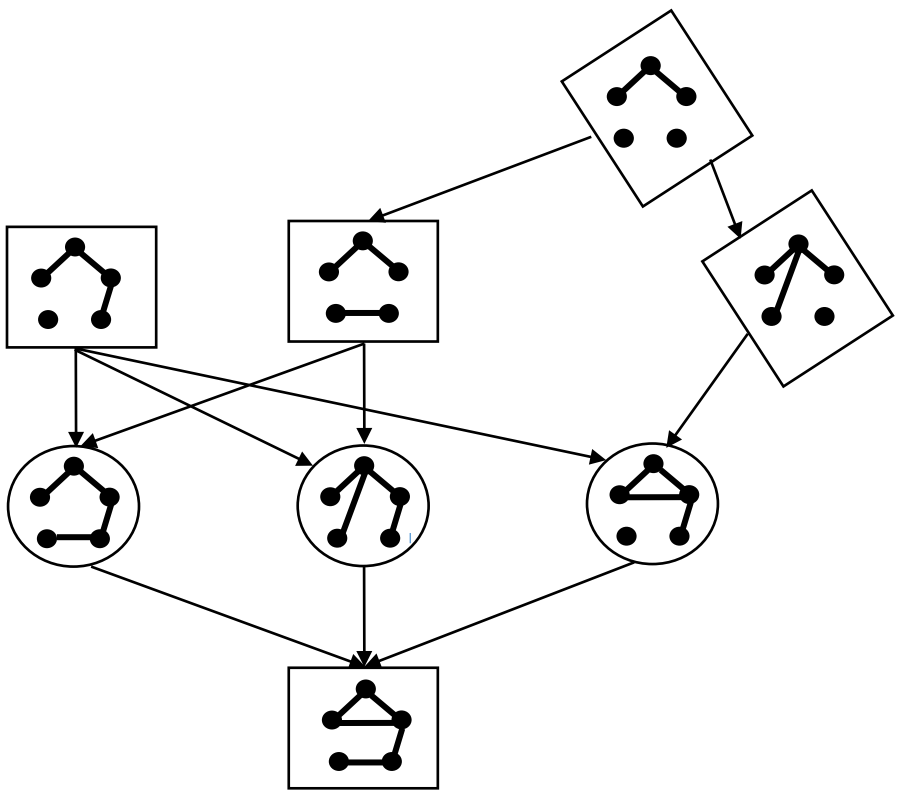

Figure 2.5.1 A nonplanar subdigraph of TD(5)

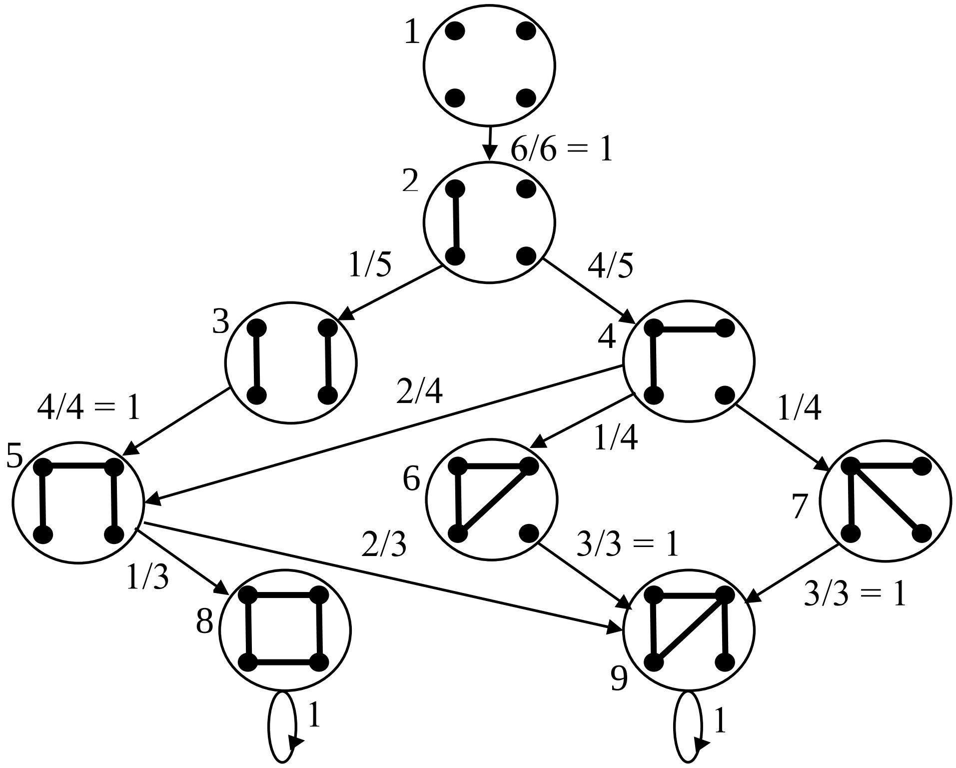

Figure 2.5.2 A planar drawing of TD(4)

(2.6) Probability distributions for graphs on a given level

Let T = [T_{ij}], where T_{ij} is the transition probability of going from G_i to G_j. This is the *transition matrix for the at most unicyclic random graph process of order n.*

Theorem 2.6.1 The probability distributions for graphs starting at the initial empty graph and terminating at level *k*, $1 \leq k \leq n$, are obtained as $T_{1j}^{(k)}$ from the first row of the *k-th* power of the transition matrix T.

Proof. The entry $T_{ij}^{(k)}$ in T^k is the probability of going from node *i* to node *j* in *k* steps. Note that in TD(*n*) motion is directed only downward from level *i* to level *i* + 1 for each edge insertion. Therefore, the set of probabilities $T_{1j}^{(k)}$ provides the probability distribution for the graphs on level *k.* In particular, the set of probabilities $T_{1j}^{(n)}$ provides the probability distribution for the terminal graphs of the at most unicyclic random graph process. ■

(2.7) Transition matrix studies

Let T denote the transition matrix for TD(4) for the at most unicyclic random graph process of order 4. Then,

$$T = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{5} & \frac{4}{5} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad T^2 = \begin{bmatrix} 0 & 0 & \frac{1}{5} & \frac{4}{5} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{3}{5} & \frac{1}{5} & \frac{1}{5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{5}{6} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T^3 = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{3}{5} & \frac{1}{5} & \frac{1}{5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{5} & \frac{4}{5} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{5}{6} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad T^4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{5} & \frac{4}{5} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{5} & \frac{4}{5} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{5}{6} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Note that for $k \geq 4$ the only non-zero probabilities that appear in T^k are those associated with the terminal graphs 8 and 9. This means that for any node in TD(4) the only directed walks of length k are those that start at the selected node and end at node 8 or node 9 and then go through the associated loop the number of times it takes to get k steps. Since the loop contributes nothing to the probability, $T^k = T^4$ for all $k \geq 4$. The preceding generalizes to a description of the transition matrix for TD(n) and in particular to the following theorem.

Theorem 2.7.1 Let T denote the transition matrix for TD(n) for the at most unicyclic random graph process of order n. Then,

$$T^k = T^n \text{ for all } k \geq n. \ \blacksquare$$

The transition matrix S for TD(5), the transition digraph for the at most unicyclic random graph process of order 5 is shown in what follows. This matrix has 19 rows and 19 columns. Its powers, as is the case for all n, will provide the probability distributions for the graphs at each level of the transition digraph TD(5).

As with TD(4) and the comments about it one can say analogously that the powers of S satisfy $S^k = S^5$ for all k $\geq$ 5 and since TD(5) has five terminal graphs, the first fourteen columns of S^5 are zero. The remaining five nonzero columns provide the probability distributions noted above.

$$S = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1/3 & 2/3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1/2 & 1/8 & 1/4 & 1/8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2/7 & 2/7 & 2/7 & 1/7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4/7 & 2/7 & 0 & 0 & 1/7 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3/7 & 3/7 & 0 & 0 & 1/7 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6/7 & 0 & 1/7 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/6 & 1/3 & 1/3 & 1/6 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/3 & 1/6 & 1/3 & 1/6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/4 & 1/2 & 1/4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

The transition matrix for the
at most unicyclic random graph process of order 5

3. Properties of DTD(n), the underlying digraph of TD(n)

$$D(4)=\begin{bmatrix}0&1&0&0&0&0&0&0&0\\0&0&1&1&0&0&0&0&0\\0&0&0&0&1&0&0&0&0\\0&0&0&0&1&1&1&0&0\\0&0&0&0&0&0&0&1&1\\0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0&1\end{bmatrix}$$

The adjacency matrix $D(4)$ for DTD(4), the underlying digraph of TD(4)

(3.1) Walks in DTD(n)

Theorem 3.1.1 (Prop. 2.5.6 **[3]**) The value of the ij-th entry in D^k, where D is the adjacency matrix of a digraph, is the number of directed walks in the digraph from node i to node j of length k. ■

$$D(4)^2=\begin{bmatrix}0&0&1&1&0&0&0&0&0\\0&0&0&0&2&1&1&0&0\\0&0&0&0&0&0&0&1&1\\0&0&0&0&0&0&0&1&3\\0&0&0&0&0&0&0&1&1\\0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0&1\end{bmatrix}\quad D(4)^3=\begin{bmatrix}0&0&0&0&2&1&1&0&0\\0&0&0&0&0&0&0&2&4\\0&0&0&0&0&0&0&1&1\\0&0&0&0&0&0&0&1&3\\0&0&0&0&0&0&0&1&1\\0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0&1\end{bmatrix}$$

$$D(4)^4=\begin{bmatrix}0&0&0&0&0&0&0&2&4\\0&0&0&0&0&0&0&2&4\\0&0&0&0&0&0&0&1&1\\0&0&0&0&0&0&0&1&3\\0&0&0&0&0&0&0&1&1\\0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0&1\end{bmatrix}\quad D(4)^5=\begin{bmatrix}0&0&0&0&0&0&0&2&4\\0&0&0&0&0&0&0&2&4\\0&0&0&0&0&0&0&1&1\\0&0&0&0&0&0&0&1&3\\0&0&0&0&0&0&0&1&1\\0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0&1\end{bmatrix}$$

The fact that $D(4)^5 = D(4)^4$ is no coincidence. First note that all directed walks regardless of where they start must end at the terminal vertices 8 or 9. For walks of length k greater than 4, the walk traverses the loop at 8 or 9 respectively until k steps are completed. This does not increase the number of walks. Thus, $D(4)^k = D(4)^4$ for all $k \geq 4$. The generalization $D(n)^k = D(n)^n$ for all $k \geq n$ is obvious.

4. Properties of GTD(n), the underlying graph of TD(n)

$$A(4) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

The adjacency matrix $A(4)$ for GTD(4), the underlying graph of TD(4)

Note that GTD(n) does not have any loops and the size of GTD(n) is the size of DTD(n) minus the number of loops in DTD(n).

(4.1) Walks in GTD(n)

Theorem 4.1.1 (Prop. 2.5.4 **[3]**) The value of the ij-th entry in A^k, where A is the adjacency matrix of a graph, is the number of walks in the graph from vertex i to vertex j of length k. ■

The first five powers of $A(4)$ are shown below. First, note that the diagonal entries in $A(4)^2$ provide the vertex degree sequence of the vertices in GTD(4). This is so because the distinct walks of length two from vertex i to vertex i coincides with the number of edges incident to vertex i.

In contrast to the powers of $D(4)$, the entries in the powers of $A(4)$ are varied with no closed form since partial to full back tracking walks produces many different walks of a given length between two vertices.

Of special interest is the fact that the underlying graph is closely related to an interesting random graph process that can be defined where both the addition and deletion of edges account for movement in the associated transition digraph. This process is called the Reversible Random Graph Process (see **[7][8]**).

$$A(4)^2 = \begin{bmatrix} 1&0&1&1&0&0&0&0&0 \\ 0&3&0&0&2&1&1&0&0 \\ 1&0&2&2&0&0&0&1&1 \\ 1&0&2&4&0&0&0&1&3 \\ 0&2&0&0&4&2&2&0&0 \\ 0&1&0&0&2&2&2&0&0 \\ 0&1&0&0&2&2&2&0&0 \\ 0&0&1&1&0&0&0&1&1 \\ 0&0&1&3&0&0&0&1&3 \end{bmatrix} \quad A(4)^3 = \begin{bmatrix} 0&3&0&0&2&1&1&0&0 \\ 3&0&5&7&0&0&0&2&4 \\ 0&5&0&0&6&3&3&0&0 \\ 0&7&0&0&10&7&7&0&0 \\ 2&0&6&10&0&0&0&4&8 \\ 1&0&3&7&0&0&0&2&6 \\ 1&0&3&7&0&0&0&2&6 \\ 0&2&0&0&4&2&2&0&0 \\ 0&4&0&0&8&6&6&0&0 \end{bmatrix}$$

$$A(4)^4 = \begin{bmatrix} 3&0&5&7&0&0&0&2&4 \\ 0&15&0&0&18&11&11&0&0 \\ 5&0&11&17&0&0&0&6&12 \\ 7&0&17&31&0&0&0&10&24 \\ 0&18&0&0&28&18&18&0&0 \\ 0&11&0&0&18&13&13&0&0 \\ 0&11&0&0&18&13&13&0&0 \\ 2&0&6&10&0&0&0&4&8 \\ 4&0&12&24&0&0&0&8&20 \end{bmatrix} \quad A(4)^5 = \begin{bmatrix} 0&15&0&0&18&11&11&0&0 \\ 15&0&33&55&0&0&0&18&40 \\ 0&33&0&0&46&29&29&0&0 \\ 0&55&0&0&82&55&55&0&0 \\ 18&0&46&82&0&0&0&28&64 \\ 11&0&29&55&0&0&0&18&44 \\ 11&0&29&55&0&0&0&18&44 \\ 0&18&0&0&28&18&18&0&0 \\ 0&40&0&0&64&44&44&0&0 \end{bmatrix}$$

(4.2) Independent cycles in GTD(n)

Given the size e and order v of the *underlying graph* GTD(n) of the transition digraph TD(n) the number IC of independent cycles of GTD(n) is obtained via the equation

$$IC = e - v + 1$$

TABLE 4.2.1 The number IC of independent cycles in GTD(n), the underlying graph of TD(n) for n = 1, 2, 3, 4, and 5

n	1	2	3	4	5
IC	0	0	0	3	17

More results on the size of GTD(n) will obviously produce information on the cycle structure of GTD(n).

(4.3) Traceability of GTD(*n*)

A path that contains every vertex of a graph is called a *Hamilton path*. A graph is called *traceable* if it contains a Hamilton path.

See **[4][5]** for comparable work on traceability related to random graph processes of the type studied here.

All one-edge transformation random graph processes have the property that the node set of the transition digraph for the process is naturally partitioned into the even sized nodes and the odd sized nodes, denoted respectively, $E(n)$ and $O(n)$.

In particular, for the at most unicylic random graph process,

$$|\mathcal{C}\ell(n)| = |E(n)| + |O(n)| = v$$

Our basic theorem for the study of traceability of GTD(*n*) is the following.

Theorem 4.3.1 If GTD(*n*) contains a Hamilton path, that is, GTD(*n*) is traceable, then

(1) GTD(*n*) has at most two pendant vertices.

(2) The Hamilton path starts (or ends) at nK_1 and ends (or starts) at C_n.

(3) $||E(n)| - |O(n)|| \leq 1$.

In particular,
$|E(n)| - |O(n)| = 0$, when $|\mathcal{C}\ell(n)|$ is even and
$|E(n)| - |O(n)| = 1$, when $|\mathcal{C}\ell(n)|$ is odd.

(4) $|\mathcal{C}\ell(n)|$ and n must have opposite parity.

(5) If a vertex in GTD(*n*) has degree 2, then the edges incident to this vertex must be in every Hamilton path in GTD(*n*).

Proof. If GTD(*n*) has a pendant (of degree 1) vertex G, then a Hamilton path in GTD(*n*) must either start or end at G. If GTD(*n*) has two pendant vertices G and H, then every Hamilton path in GTD(*n*) must start and end at these two vertices. If there is a third pendant vertex, no Hamilton path could cover it and GTD(*n*) would not be traceable. Thus, (1) if GTD(*n*) is traceable, it contains at most two pendant vertices.

The vertices nK_1 and C_n are pendant vertices in traceable GTD(*n*). Thus, (2).

Without loss of generality, let a Hamilton path in GTD(n) start at nK_1 (vertex of even size), then as one progresses along this path the vertices of the path alternate between even and odd size vertices. If $||E(n)| - |O(n)|| > 1$, the path will get to a point where 2 or more vertices of the same parity would be left to cover and this would not be possible due to the alternate parity requirement. Thus, (3) $||E(n)| - |O(n)|| \leq 1$ and clearly, $|E(n)| - |O(n)| = 0$, when $|\mathcal{C}\ell(n)|$ is even and $|E(n)| - |O(n)| = 1$, when $|\mathcal{C}\ell(n)|$ is odd.

A Hamilton path starting at nK_1, if it exists, must end at C_n with n odd when $|\mathcal{C}\ell(n)|$ is even and must end at C_n with n even when $|\mathcal{C}\ell(n)|$ is odd. This is seen by following the alternate parity of the vertices in the path. Thus, (4) $|\mathcal{C}\ell(n)|$ and n must have opposite parity.

For a vertex in GTD(n) having degree 2 to be covered by a Hamilton path it is necessary that the two incident edges to this vertex must also be in the Hamilton path. Therefore, such a pair of edges must be in every Hamilton path in GTD(n). ■

Theorem 4.3.2 GTD(n) is traceable for $n = 1$, 2, and 3 and not traceable for $n = 4$, 5, 7, 12, 13, and 14.

Proof. For $n = 1$, 2, and 3, see **Figure 2.4.1**. For $n = 4$, see **Figure 2.5.2** and note that there are two vertices of degree 2 in GTD(4) whose incident edges form a 4-cycle. By (5) in **Theorem 4.3.1** these four edges must be in every Hamilton path. Thus, GTD(4) cannot be traceable since a path cannot contain a 4-cycle. For $n = 5$, the order $|\mathcal{C}\ell(5)| = 19$ has the same parity as 5, thus by (4) in **Theorem 4.3.1** GTD(5) is not traceable. Further examining the parity relation of n and the order of $\mathcal{C}\ell(n)$ yields a number of special cases where, by (4) in **Theorem 3.4.1** will yield GTD(n) is not traceable for $n = 7$, 12, 13, and 14 (see **Table 2.2.1**). ■

Comment A look at the known extensions of **Table 2.2.1** using (4) of **Theorem 4.3.1** will yield additional values of n for which GTD(n) is not traceable.

Problem 4.1 Find additional theorems that will determine the traceability or non-traceability of GTD(n).

5. Concluding remarks

The work done here is in the spirit of the research on random graph processes based on one-edge transformations of graphs that was done in depth in **[6]**. Such research leads to questions about enumeration and the structure of graphs. The solutions to these questions require a combination of theoretical and computer implementation techniques. Although not investigated here, there are interesting applications that can be followed up on. Two such examples are the study of biological and chemical networks **[9]-[11][12]-[13]**. The basic structural evolution in random graph processes is a natural inducement to seek applications.

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