A288932 = 1,0,1,0,1,1,0,1,0,1,1,1,0,1,0,...

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Proof of A288932(n+1) = A308185(n)

Here is a proof of Neil Sloane's conjecture in A308185 that $a(n+1) = \mathrm{A308185}(n)$ for $n \geq 1$.

First we have to find a mathematical way to generate $(a(n)) = \mathrm{A288932}$, which is created as a fixed point of the StringReplace procedure SR in Mathematica. In the case of A288932 by

$$\mathrm{SR}(00) = 1000 \quad \mathrm{SR}(10) = 10101.$$

Note that we can ignore the production $00 \mapsto 1000$, because it only occurs only at the end of the iterates $\mathrm{SR}^n(00)$: its influence disappears in the limit as n tends to infinity. So we will consider the iterates $\mathrm{SR}^n(10)$:
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$10101 = \mathrm{SR}(10)$
$10101101011 = \mathrm{SR}^2(10)$, etc.
Note that these converge to $(a(n))$.

Let β be the block substitution on the set of words $\{10, E\}^*$ over the alphabet $\{0, 1, E\}$ given by

$$\beta(10) = 1010E, \quad \beta(E) = E.$$

Note that this block substitution and its iterates are well defined on $\{10, E\}^*$.

Let λ be the letter to letter substitution given by $\lambda(0) = 0$, $\lambda(1) = 1$, $\lambda(E) = 1$.

CLAIM 1: $\mathrm{SR}^n(10) = \lambda(\beta^n(10))$ for $n \geq 1$.

Proof: By induction. This is true for $n = 1$. Suppose true for n. Then

$$\begin{aligned}
\mathrm{SR}^{n+1}(10) &= \mathrm{SR}^n(10101) = \mathrm{SR}^n(10)\mathrm{SR}^n(10)1 \\
&= \lambda(\beta^n(10))\lambda(\beta^n(10))\lambda(E) \\
&= \lambda(\beta^n(1010E)) \\
&= \lambda(\beta^{n+1}(10)).
\end{aligned}$$

Let μ be the morphism on $\{0, 1\}^*$ given by $\mu(0) = 0101, \mu(1) = 1$. The infinite fixed point of μ is the sequence A308185, by definition.

CLAIM 2: $\lambda(\beta^n(10)) = 1\,\mu^n(0)$ for $n \geq 1$.

Proof: By induction. For $n = 1$ one has $\lambda(\beta(10)) = \lambda(1010E) = 10101 = 1\,\mu(0)$.

Suppose it holds for n. Then

$$\begin{aligned}
\lambda(\beta^{n+1}(10)) &= \lambda(\beta^n(1010E)) \\
&= \lambda(\beta^n(10)\beta^n(10)\beta^n(E))) \\
&= 1\,\mu^n(0)1\,\mu^n(0)\,1 \\
&= 1\,\mu^n(01011) \\
&= 1\,\mu^{n+1}(01).
\end{aligned}$$

Combining CLAIM 1 with CLAIM 2, one obtains $a(n+1) = \mathrm{A308185}(n)$ for $n \geq 1$.