A282939: Maximum number of straight lines required to draw the boundary of any polyomino with n squares.

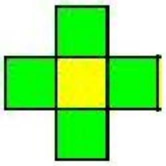
It is conjectured that a(n) = 2*n+2 for n (mod 4) == 1, otherwise a(n) = 2*n.

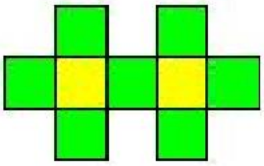
A proof of the conjecture follows, but it is useful first to note two details.

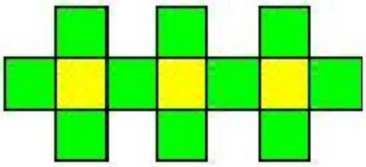
1. As is well known, a polyomino with perimeter 2*n+2 is a tree. Hence any polyomino requiring 2*n+2 straight lines for drawing its border is also a tree.
2. For any polyomino requiring 2*n+2 straight lines for drawing its border, said border will coincide with the edges of squares that are all of the same colour, if chessboard colouring is applied to the polyomino. For clarity, see the diagrams in first point of the proof below.

Proof of conjecture.

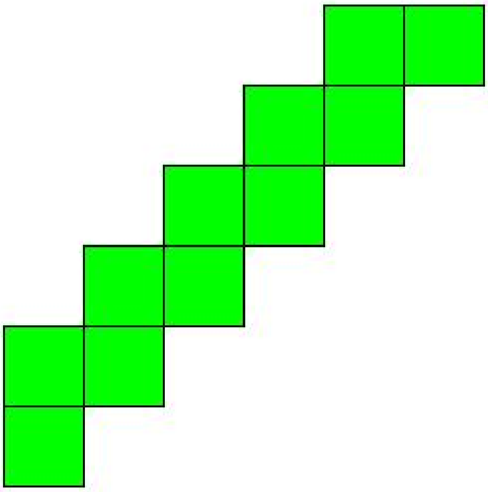
1. a(n) = 2*n+2 for n (mod 4) == 1. Certainly a(n) is not greater than 2*n+2, as this is the maximum perimeter. a(n) is certainly at least 2*n+2, for n (mod 4) == 1, as shown by repeatedly adding airplane tetrominoes to a mononimo.







2. a(n) is at least 2*n for any n, including for n (mod 4) != 1, as shown by building a staircase:



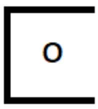
3. It is now sufficient to prove that, if the number of straight lines needed for drawing a polyomino is equal to its perimeter, and this perimeter is maximal (that is 2*n+2), then the polyomino can be made by repeatedly adding airplane tetrominoes to a monomino, and so has size n such that n (mod 4) == 1. As a consequence, a(n) = 2*n for n (mod 4) != 1.

   Proof: Certainly no polyomino not so composed of airplanes but having the number of straight lines needed for drawing it equal to its perimeter, has been identified.

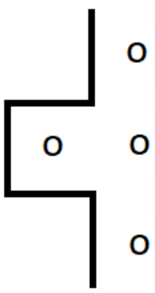
   Suppose one does exist; take as an example such a polyomino of the smallest size possible.

   Choose, within said polyomino, the lowest of the leftmost cells.

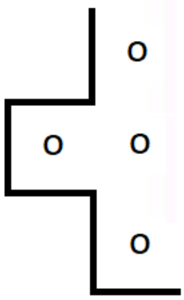
   Draw the cell, and use an o to symbolise an occupied cell, an x for an empty cell, where not obvious.



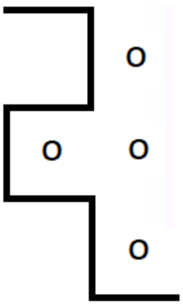
   It is now possible to deduce some nearby cells and border line segments. Remember always that at no point of the boundary is there a line segment of 2 unit lengths.



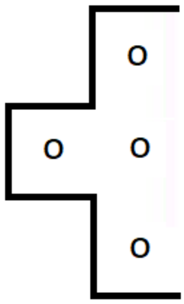
We also know that the cell on the left is the lowest leftmost cell of the polyomino, so we can also deduce:



Examine then possibilities 3a and 3b, that are the two alternatives that derive from the above figure:

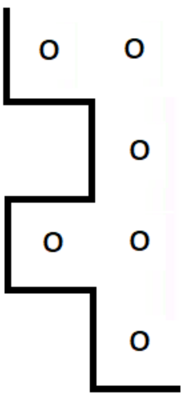


3a

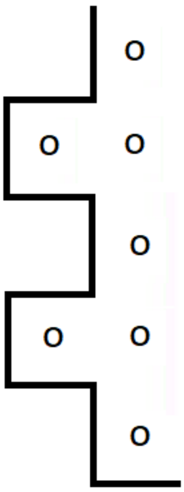


3b

From 3a we can deduce:

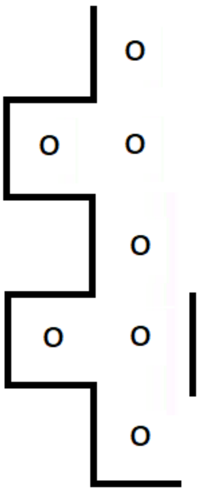


And hence:

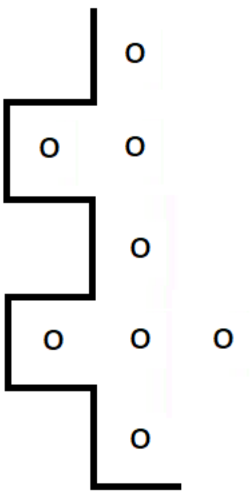


If we apply chessboard colouring to the polyomino, it is clear that the external border will have line segments that coincide only with the edges of cells with all the same colour.

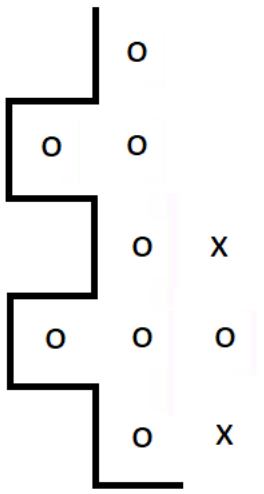
Therefore the isolated segment marked below cannot be part of the border:



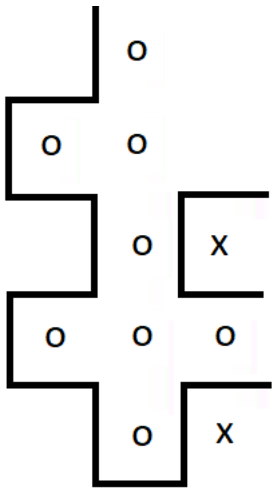
Therefore:



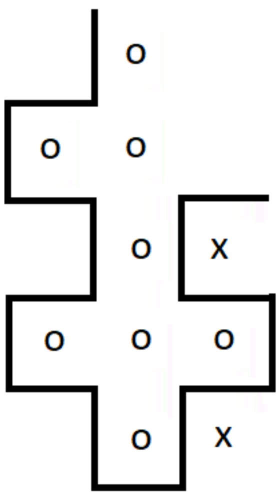
There cannot be a 2 by 2 square of occupied cells in a tree, so:



So:

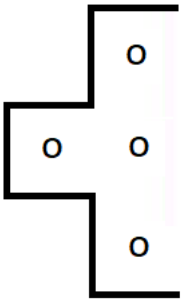


So, as the upper x cannot be in a hole::



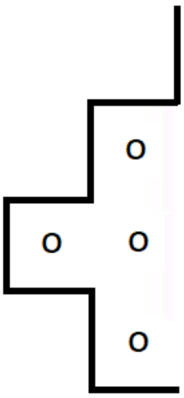
But by removing the airplane at the bottom, we could obtain a smaller polyomino satisfying the input criteria, contradicting the assumption that this one was minimal.

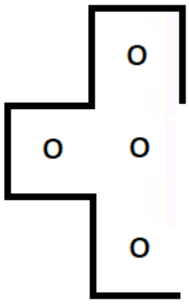
Try then 3b, which we will reproduce for convenience:



3b

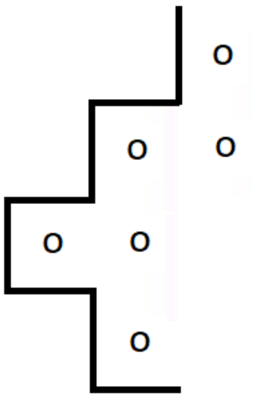
3b can give rise to 3c or 3d:



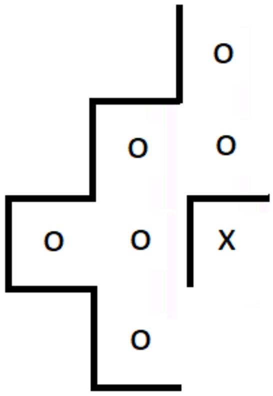


3c 3d

From 3c we can deduce:

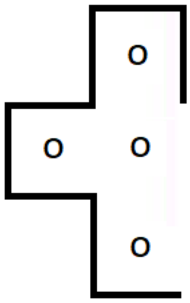


And hence:



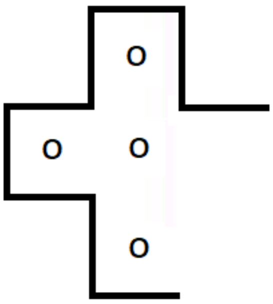
But remembering that the border must be composed of line segments that coincide with squares of all the same colour if chessboard colouring is applied to the polyomino, the segment to the left of the x above leads to a contradiction.

There remains 3d, which we will reproduce for convenience:

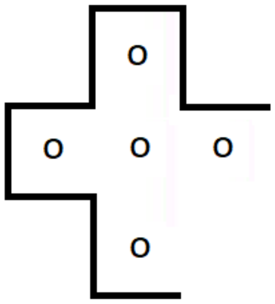


3d

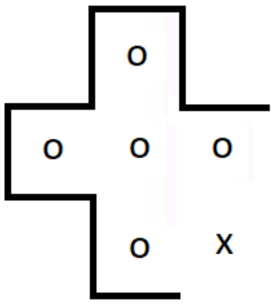
Leads to:



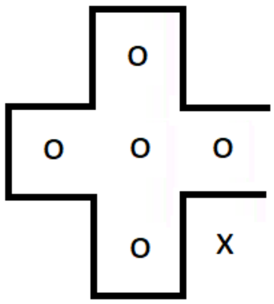
And so:



And, as there can't be a 2 by 2 square:



And so:



But by removing the airplane at the left, we could obtain a smaller polyomino satisfying the input criteria, contradicting the assumption that this one was minimal.

All the cases examined lead to contradictions and therefore the proof is complete.

John Mason, June 2021