Fibonachos

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January 28, 2017

Theorem 1. *The smallest number that requires n rounds of Fibonachos is $F(2n+1)-n$.*

Proof. The sum of the first n Fibonacci numbers is $F(n+1)-1$ (A000071). Clearly, every member of A000071 requires one round. Every other number j can be written as $j = F(k)-m$ for some k where $F(k) \leq j$ and $m > 1$. One round of Fibonachos will remove $F(k-1)-1$ nachos, reducing j to $F(k)-m-(F(k-1)-1) = F(k-2)-m+1$.

Now, suppose inductively that $F(2n-1)-(n-1)$ is the smallest number requiring $n-1$ rounds. Let $F(2n-1)-n-1 < j \leq F(2n+1)-n$. Then, write $j = F(k)-m$ as above. We must have $2n-1 \leq k \leq 2n+1$. After one round, we are left with $F(k-2)-m+1$ nachos. If $j = F(2n+1)-n$, then $k = 2n+1$ and $m = n$, which makes this value $F(2n-1)-(n-1)$. So, this value of j requires n rounds, as after 1 round it requires $n-1$ rounds. If $j < F(2n+1)-n$, then the result after one round is less than $F(2n-1)-(n-1)$. So, this value of j requires fewer than n rounds, as after 1 round it requires fewer than $n-1$ rounds. □

Theorem 2. *A280523(n) = A215004($2n-2$)*

Proof. A215004 is defined by the recurrence $a(n) = a(n-1)+a(n-2)+\left\lfloor \frac{n}{2} \right\rfloor$. We wish to show that A215004(n) $= F(n+3) - \left\lfloor \frac{n+3}{2} \right\rfloor$. This would cause A215004($2n-2$) $= F(2n+1) - \left\lfloor \frac{2n+1}{2} \right\rfloor = F(2n+1)-n =$A280523($n$), as required.

We have A215004(0) $= 1 = 2-1 = F(3) - \left\lfloor \frac{3}{2} \right\rfloor$ and A215004(1) $= 1 = 3-2 = F(4) - \left\lfloor \frac{4}{2} \right\rfloor$. Inductively, suppose A215004(k) $= F(k+3) - \left\lfloor \frac{k+3}{2} \right\rfloor$ for $k < n$. Then,

$$\begin{aligned} \text{A215004}(n) &= \text{A215004}(n-1) + \text{A215004}(n-2) + \left\lfloor \frac{n}{2} \right\rfloor \\ &= F(n+2) - \left\lfloor \frac{n+2}{2} \right\rfloor + F(n+1) - \left\lfloor \frac{n+1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor \\ &= F(n+3) + \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n+1}{2} \right\rfloor - \left\lfloor \frac{n+2}{2} \right\rfloor . \end{aligned}$$

If n is even, then this is

$$\begin{aligned} F(n+3) + \frac{n}{2} - \frac{n}{2} - \frac{n+2}{2} &= F(n+3) - \frac{n+2}{2} \\ &= F(n+3) - \left\lfloor \frac{n+3}{2} \right\rfloor , \end{aligned}$$

as required.

If n is odd, we instead have

$$F(n+3) + \frac{n-1}{2} - \frac{n+1}{2} - \frac{n+1}{2} = F(n+3) - \frac{n+1}{2} - 1$$
$$= F(n+3) - \left\lfloor \frac{n+3}{2} \right\rfloor,$$

as required. □

Here is an alternative proof:

Proof. A215004 is defined by the recurrence $a(n) = a(n-1) + a(n-2) + \lfloor \frac{n}{2} \rfloor$. This is alternatively written as $a(n) = a(n-1) + a(n-2) + \frac{n}{2} + \frac{(-1)^n}{4} - \frac{1}{4}$. This is a nonhomogenous linear recurrence which can be solved by the Method of Undetermined Coefficients. Doing so (e.g. with Maple) yields a closed form for $a(n)$. Evaluating this closed form at $2n-2$ gives a closed form for $a(2n-2)$, and this is the same formula as the closed form for $F(2n+1) - n$. □

Looking at the entry for A215004, I think Theorem 2 was already known in some capacity.