The Binary Operation Property X(YZ) = XZ

by David Pasino

People reading this article in future times will almost surely have different math training than we had. This writer, in 2025, trained in the Twentieth Century, will use rudimentary language of semigroups and semigroup actions on sets, to study binary operations on a set that are closed and have the property x(yz) = xz identically. A **semigroup** is an abstract or concrete algebra system of one binary operation that is closed on its operand domain and has the associative property x(yz) = (xy)z identically. Addition or multiplication of positive integers are prototypical examples of semigroups. Compound products in a semigroup, as in a group, need no bracketing to channel them down to a single value, but, unlike elements in a group, elements in a semigroup need not have an inverse, and a semigroup need not even have an identity element.

Semigroup literature is vast.

A binary operation, in this article, is a function whose domain is a full product of two sets, which may be the same set. The elements of a product set are ordered pairs, and the function value of a binary operation at an ordered pair may be denoted by a symbol or dot between the operands, or mere juxtaposition of the operands, for visual ease when clarity permits. When function notation F(x) is used for value of function F at domain element x, and ordered pair notation (x, y) is in use, then the value of a binary operation B at (x, y) could be written as x B y or B((x, y)), but we customarily shorten those to xy and B(x, y) when the meaning remains clear.

A binary operation whose domain is the product of some set with the same set is called a **closed** operation if its function values are also in that same set. A set together with a closed binary operation on it is called a **magma**.

A **homomorphism** from a magma (M, B) to a magma (N, C) is a function F: M → N such that F(x B y) = F(x) C F(y) for all x and y in M.

Functions on a set S into itself are called **endofunctions** of S, and the set of them, with binary operation of composition of functions, is denoted by **End(S)**.

Function composition order can be reversed by writing functions on the right of their arguments, but in this article we write functions on the left, and use ∘ as composition symbol, as in $f\big(g(x)\big) = (f \circ g)(x)$. We take End(S) to be a magma by this operation ∘.

A homomorphism of a magma M into End(S) for a set S is, thus, by intended unravelment of definitions, an assignment of an endofunction of S to every element of M, call it $\theta_m \in End(S)$ for $m \in M$, such that $\theta_{mn} = \theta_m \circ \theta_n$ for all m and n in M. From this we obtain a binary operation of the form $M \times S \to S$, by $m \cdot s = \theta_m(s)$. Since composition of functions is associative, End(S) is a semigroup, and so is its submagma $\theta^{>}(M) = \{\theta_m : m \in M\}$, which is a quotient of M. But that's a generality we're just giving that glance to, in assuming M itself is already associative. Associativity of M implies $m \cdot (n \cdot s) = \theta_m\big(\theta_n(s)\big) = (\theta_m \circ \theta_n)(s) = \theta_{mn}(s) = mn \cdot s$. Conversely, if · were given as a function $M \times S \to S$ satisfying $m \cdot (n \cdot s) = mn \cdot s$ identically, and $\theta: M \to End(S)$ were defined by $\theta_m(s) = m \cdot s$, then θ would be a homomorphism.

A binary operation of the form $M \times S \to S$, for a semigroup M and a set S, satisfying $m \cdot (n \cdot h) = mn \cdot h$ identically, is called a **left action** of M on S.

A major tool in semigroup theory since the 1950s that we will not use in this article but which the writer has read about and should later examine for what we do here is called Green's Relations. I am not yet enough of a semigroup theorist to tell you what they are, but I can at least tell you what they're called, Green's Relations in a semigroup.

We are now going to specialize way down to study the actions of one of the plainest kinds of semigroup. On any set M, the left projection binary operation, xy = x for all x and y in M, is associative, as (xy)z = xy = x = x(yz). So a mere set M is a semigroup by wearing its left projection binary operation, and we can look at what our fancy language setup for semigroup actions says in this very plain case. The action condition $m \cdot (n \cdot s) = mn \cdot s$, with $mn = m$ for left projection, says $m \cdot (n \cdot s) = m \cdot s$. When the set that the semigroup M is acting on in this way is M itself, then the action is a closed binary operation, satisfying the identity x(yz) = xz.

Any semigroup action M × S → S determines a partition of the set S, by the equivalence closure of the relation $s \sim t \Leftrightarrow (\exists m \in M)(ms = t)$ in S. If M is left projection, ◀, then observe that if h and k are in S such $m \bullet h = m \bullet k$ for one m in M, then $n \bullet h = n \bullet k$ for every n in M, since $n \bullet h = (n ◀ m) \bullet h = n \bullet (m \bullet h) = n \bullet (m \bullet k) = (n ◀ m) \bullet k = n \bullet k$. In other words, the same relation of elements h and k in S is defined by the condition

$(\exists m \in M)(m \bullet h = n \bullet k)$

as by the condition

$(\forall m \in M)(m \bullet h = n \bullet k)$,

for • a left action by ◀.

This relation is reflexive, as $m \bullet h = m \bullet h$;
symmetric, as $m \bullet h = m \bullet k \Longrightarrow m \bullet k = m \bullet h$;
and transitive, for if $(n, m, t, k, h) \in M \times M \times S \times S \times S$ such that
$n \bullet t = n \bullet k$ and $m \bullet k = m \bullet h$, then $n \bullet k = n \bullet h$ so $n \bullet t = n \bullet h$.
Let Q be the partition associated with this equivalence relation on S.
For each n in M, the action function $n \bullet$ is constant on blocks of the partition Q, and closed on each block, by the definition of the equivalence. Therefore the collective image $n \bullet S$ is a transversal of Q. Thus, in the said setup, the semigroup action induces a transversal-valued function, to be called G.

A converse holds as well, that if a set M is having its elements mapped to subsets of a partitioned set S by a function F such that each value F(n) is a transversal of the partition, then this mapping induces a left action of (M, ◀) on S. To prove this, denote the given partition by P, and denote also by P the mapping of elements of S to their blocks in P. Then define the semigroup action function by $m \bullet s \in F(m) \cap P(s)$. This is an action, as

$(m \blacktriangleleft n) \bullet s = m \bullet s \in F(m) \cap P(s),$
$m \bullet (n \bullet s) \in F(m) \cap P(n \bullet s) = F(m) \cap P(s),$

that last equation because $P(n \bullet s) = P\left(elt\big(F(n) \cap P(s)\big)\right) = P(s)$.

These passages between the setup of a left action of (M, ◀) on S and the setup of a partitioned set (S, P) and a mapping of elements of M to transversals of P are mutually inverse, so these two setups are different forms of the same information.

Proving that passing from either form to the other and then back to the first form gets back to the same instance of the first form seems tedious to me. It might be more interesting to prove in discussion than just a person saying it. It's important, so here's my two cents. (The acronym TVF will refer to the transversal-valued function form of the data. Letters standing for data items will be as said just above.)

Q is the level partition of the map $S \to S^M$ that sends each $s \in S$ to $(\bullet\ s)\colon M \to S$.

In passing from (S, P, M, F) to (S, M, •) and on to (S, Q, M, G), the equivalence relation associated with P is the same as the equivalence relation associated with Q, the relation of having the same effect as factors in the action. Hence Q = P. Also G = F, as, for each $m \in M$,

$$G(m) = m \bullet S = \{m \bullet s : s \in S\} = \{elt\big(F(m) \cap P(s)\big) : s \in S\}$$
$$= \bigcup_{s \in S} F(m) \cap P(s) = F(m) \cap \bigcup_{s \in S} P(s) = F(m) \cap S = F(m).$$

Hence in passing from TVF to magma and on to TVF, it's the same TVF. Also, no two different actions $M \times S \to S$ have the same map $S \to S^M$, but in a sequence of passages between the magma form and the TVF form, the partitions induced and received are always equal.

OK? It's not difficult, it's just tedious for a person alone.

It's like a 2-valued spin; where it goes to is where it comes from.

A left action of (M, ◀) on S is thus "the same thing" as a partition of S and a function on M into the power set of S such that the function values are transversals of the partition. This gives us a way to count the actions. Let m be the number of elements of M, and n the number of elements of S. Summing on partitions of the number n, in the format $n = 1t_1 + 2t_2 + \cdots + nt_n = t^+$, that is t_r the number of parts of size r in the partition t, the number of partitions of S of shape t is $\frac{n!}{\prod_{r=1}^{n} (r!)^{t_r} \cdot t_r!}$, and each of those has $\prod_{r=1}^{n} r^{t_r}$ transversals, so the number of actions is $\sum_{t^+=n} \frac{n!}{\prod_{r=1}^{n}(r!)^{t_r} \cdot t_r!} \cdot \left(\prod_{r=1}^{n} r^{t_r}\right)^m = \sum_{t^+=n} n! \prod_{r=1}^{n} \frac{r^{mt_r}}{(r!)^{t_r} \cdot t_r!} = \sum_{t^+=n} \prod_{r=1}^{n} \frac{r^{1+mt_r}}{(r!)^{t_r} \cdot t_r!} = \sum_{t^+=n} \prod_{r=1}^{n} \frac{r^{mt_r}}{((r-1)!)^{t_r} \cdot t_r!}$.

When m = n, this gives the number of closed binary operations of the variety x(yz) = xz on n elements,

n	
0	1
1	1
2	5
3	52
4	1445
5	104116
6	16379797
7	6067246144
8	5270005429705
9	9832425683734288
10	40944833826904310921
11	384044953998005246634304
12	7656468877618298485395299533

This is number sequence is A279644, so this note is written to be offered there. A natural companion question is what are the right actions of a left projection like. A **right action** of a semigroup M on a set S is a function $S \times M \to S$ such that $s \cdot (mn) = (s \cdot m) \cdot n$ for all s in S and all m and n in M. So a right action of a left projection satisfies $s \cdot m = (s \cdot m) \cdot n$. This says $Ran(\cdot\, m) \subseteq Fix(\cdot\, n)$, holding for all m and n in M, and since the fixed point set of a function is a subset of the range, this means that all the functions $(\cdot\, n)$ are idempotent and have the same range. Taking letter n again to be a number, the number of those functions on n elements is $\sum_{r=1}^{n} \binom{n}{r} r^{n-r}$, so the number of right actions of a left projection semigroup of m elements on a set of n elements is $\sum_{r=1}^{n} \binom{n}{r} r^{(n-r)m}$. When m = n, this sequence starts as follows.

n	right actions of n-element left projection semigroup on self set
0	0
1	1
2	3

3	28
4	1865
5	923296
6	8251820017
7	955469094105688
8	197544717126474992652 9
9	16269865048227852170384077100 8
10	301823161128486957889699208909150007041
11	1380019914148995006222290994928016517872906823270 4
12	441509087462963366728062400113220791193893345977244935586140 17

The full arrays start out as, for left actions of an m-element left projection semigroup on a set of n elements,

1	1	2	5	15	52	203	877	4140
1	1	3	10	41	196	1057	6322	41393
1	1	5	22	125	836	6277	52396	479593
1	1	9	52	413	3916	41077	481384	6198425
1	1	17	130	1445	19676	288517	4768240	86825545
1	1	33	340	5261	104116	2133397	49873552	1290339353
1	1	65	922	19685	572036	16379797	542459296	20028938953
1	1	129	2572	75053	3228316	129317797	6067246144	321040274585
1	1	257	7330	289925	18565676	1042651237	69221777920	5270005429705

and for right actions,

0	1	3	7	15	31	63	127	255
0	1	3	10	41	196	1057	6322	41393
0	1	3	16	137	1536	22417	407884	8920641
0	1	3	28	497	12736	517297	28793248	2095968065
0	1	3	52	1865	107856	12598657	2124070096	510096322689
0	1	3	100	7121	923296	318454177	160528193512	126154718406593
0	1	3	196	27497	7956336	8251820017	12317891320624	31448759910116481
0	1	3	388	107057	68883136	217260596497	955469094105688	7872626328563808065
0	1	3	772	419465	598567056	5777442092257	747336248963311136	1975447171264749926529

A left action by the empty set is just a partition with 0 transversals selected, and a right action is just a nonempty subset to be in the role of fixed points for 0 idempotent functions.

The sequence of left action counts of (M, ◀) on M has only been in OEIS since 2016, and that meaning of it has only been there since 2023. The sequence of right action counts of (M, ◀) on M was not in OIES when I checked just now. The right action count sequence is new to me in doing this write-up on the left action count. I've been studying the binary operation property x(yz) = xz since 1989, originally defined in the TVF way, abstracted from the "fact of life" of everyone's having one parent of each sex, the parents being the value of a transversal-valued function on the set of people relative to the partition by sexes; the magma product xy is the parent of x of the same sex as y, but I think this occasion is my first time studying the right actions of the left projection.

I'd like to ask how else the count of binary operations satisfying x(yz) = xz as an identity was made than by how to pick a transversal for every element in a partitioned set, how else did the property x(yz) = xz come up than as algebra of sexes and parents? And I'd like to offer other examples I've studied, of that kind of binary operation or action, and 2 words I made to refer to them by. My interest in these has been deep for a long time. I wrote proofs in a long paper submitted to a journal last year and was told it was not generally interesting to mathematicians. Also, in the 1990s, I could not get anyone to grant or critique the premise that sexes are a partition and having definite parents is a transversal-valued function in the partition. Not even my own mother, who seemed like the person I'd mostly gotten the idea from of where babies come from, how my parents had chosen people to be grandparents for us kids. Everyone seemed to deem it too obviously true to give assent on, and bizarre to seek assent on. But a glimmer of interest besides mine in this math is apparent in A279644, recognition by Andrew Howroyd that x(yz) = xz is a binary operation property to be noted and reckoned with. I recognize the name of Andrew Howroyd. He is an OEIS editor, decider of whether to admit this note into A279644. I've had some great world-class ideas and wrote them up well but was too much of a nobody to get them published but I'd still like to partner with Andrew and friends in trotting some of these things out without proof here and give other people a chance to publish proofs of them ahead of me.

§ Another kind of magma than that of people satisfying x(yz) = xz is an indexed, oriented rectangular array, with the product of points (a, b) and (c, d) being one of the endpoints of the other diagonal of the rectangle that (a, b) and (c, d) are the endpoints of one diagonal of, product coherently oriented everywhere in that (a, b)(c, d) is either (a, d) for all point pairs or (c, b) for all point pairs. This kind of magma is called a **rectangular band**, and besides satisfying x(yz) = xz, satisfies x(yz) = (xy)z.

The most general kind of magma that satisfies both x(yz) = xz and x(yz) = (xy)z is a rectangular band with a cloud of square roots adjoined to every element in the band, and products of adjoined elements x and y in any one or two of the clouds defined by xy = (xx)(yy) which follows from the two assumed properties by xy = x(xy) = (xx)y = (xx)(yy), and product of "squares" is defined in the band.

Deindexing the array leads to the number of associative operations satisfying also x(yz) = xz, on n elements,

$$A(n) = \sum_{k=1}^{n} \binom{n}{k} k^{n-k} k! \sum_{rs=k} \frac{1}{r!\,s!}$$

1	1
2	4
3	17

4	84
5	507
6	3668
7	31117
8	312938
9	3669671
10	48071832
11	690121137
12	10912283462
13	189568637611
14	3551946948020
15	70691032612277
16	1495186187390130
17	33904761111198159
18	825492020819021552
19	21358930280058590041
20	580041780169963055582

and the number of them that are plain rectangular bands without extra square roots,

$$B(k) = k! \sum_{st=k} \frac{1}{s!\,t!}.$$

1	1
2	2
3	2
4	8
5	2
6	122
7	2
8	1682
9	10082
10	30242
11	2
12	7318082
13	2

14	17297282
15	3632428802
16	36843206402
17	2
18	2981705126402
19	2
20	1690185726028802

This is A121860.

A rectangular band is the direct product of a left projection and a right projection.

A word I made up to refer to the mathematical structure of partitioned set with transversal-valued function, or magma satisfying x(yz) = xz identically, was familoid, since oid was a popular ending for words meaning particular kinds of math magmas, and family was a key idea in the original main motivating example from which the math structure was discerned to be abstracted into the definition of TVF structure, which then was soon found equivalent to x(yz) = xz structure, and noticed to be semigroup action by a trivial sort of semigroup which I knew to be the identity element of a binary operation on binary operations by x AB y = (x B y) A (y B x), so this setup remined me a little of the tangent vectors to the identity of an analytic group, here the actions of the identity element of a combinatorial group of binary operations, and rectangular bands were global product sets, while animal familoids would fit in the local product seat where fiber bundles sit in the tangent space picture.

In any familoid F, the subset consisting of all the parents is the pointwise product FF, and it is a subfamiloid of F.

In an associative familoid, every parent is its own square, the set of parents is equal to the set of squares, and all elements participate only through their squares, xy = x(xy) = (xx)y = (xx)(yy), by the familoid, associative, and familoid properties.

$$(xy)^2 = (xy)(xy) = (xy)y = x(yy) = x\big(x(yy)\big) = (xx)(yy) = x^2y^2$$

Squaring is an endomorphism in an associative familoid.

§ Every set of more than 1 elements has two extreme partitions, the coarsest and the finest. A transversal of the coarsest partition is given by a single element, so a transversal-valued function in that case is given by an ordinary endofunction. The finest partition has only the whole set as its only transversal, so a set has only one familoid on it with finest partition. The number on familoids on n elements with extreme partition is $n^n + 1$.

§ An element X in a familoid has a number of formal bracketed powers of degree n, the Catalan number $C(n) = \binom{2n-2}{n-1}/n$. Let $H(k,n)$ the number of them that reduce to degree k by doing all the reductions availed by the familoid property x(yz) = xz. Then

$$H(k,n) = \sum_{j=1}^{[k/2]} (-1)^{j-1} \binom{k-j-1}{j-1} C(n-j)$$

which implies

$$\lim_{n\to\infty} \frac{H(k,n)}{C(n)} = \frac{k-1}{2^k}$$

and the identity

$$C(n) = \sum_{j=1}^{[n/2]} (-1)^{j-1} \binom{n-j}{j} C(n-j)\,,$$

which later went into making a formula for the number of partial bracketings of an operand string toward evaluability by general binary operations,

$$G(n) = \frac{1}{2^{n+1}} \sum_{k=1}^{[n/2]} C(n-k)(-1)^{k-1} \binom{n-k}{k} (3^{2k}-1)3^{n-2k}$$

which is A001003.

It took me 50 pages of good work with infinite series and recursive arrays to do all that with C, H, G, and I guess that's off-putting, but it included a few pages of trying to make the low-degree cases accessible for children to enjoy doing algebra with powers of themselves and charting the distribution of how the powers break down into their mother and grandmother and great-grandmother and so on, or their father and grandfather and so on, or who a monomial of them and their friends would work out to.

§ I venture to use the word "ideal" from Ring Theory in talking about magmas, borrowing just the strong closure property of ideals in rings, to mean by **a left ideal in a magma**, a subset X ⊆ M for which merely the simple bare collective pointwise product inclusion MX ⊆ X holds, and a **right ideal** in a magma M is a subset X satisfying XM ⊆ X.

So, a familoid is a magma, and a magma has left and right ideals. What else can we say about left and right ideal subsets of a familoid of people?

Right ideals are closed under ancestry: if R is a right ideal in a familoid F, and x ∈ R, and y is a parent of x, then y = xz for some z ∈ F (at least y = xy), so y = xz ∈ R since x ∈ R and R is a right ideal.

Races are right ideals.

Sexes are left ideals, since xy always has the same sex as y. A left ideal is a set that contains every parent in every sex that it contains anyone at all in. So, for each sex S, a left ideal L either contains all of FS or is disjoint from S. The possible intersections of L with S are the empty set and the supersets of FS in S.

On a familoid of m elements, with sex partition block sizes

$$m = \sum_{r=1}^{k} m_r$$

and parent counts

$$n = \sum_{r=1}^{k} n_r$$

in the sexes, the number of left ideals is

$$\prod_{r=1}^{k} 2^{m_r - n_r + 1} = 2^{\sum_{r=1}^{k} m_r - n_r + 1} = 2^{m-n+k}.$$

In a familoid with m elements, k sexes, and n parent elements, the number of left ideals is 2^{m+k-n}.

§ Another word I made up to mean the same as familoid but downplaying the animal nature origin of the idea was skif, made from skip via skiph to describe something about the property x(yz) = xz, it skips the middle operand in right-bracketed strings of 3 operands. That word is also for a kind of rowboat, but that's outside my specialty. The following paragraph is lifted from the aforementioned long paper declined by a journal and has my math word skif in it.

Suppose (F, R, T) is a familoid (meaning F is a set, R is a partition of F, and T is a function on F such that T(x) is a transversal of R for every x in F), and suppose also that • is a left action of the semigroup $(F,◀)$ on a set S, and Y is a function on S into F, such that Y is into every piece of R. Let the pieces (blocks) of R be labeled. Partition S by the same labels so that each $s \in S$ gets the same label as $Y(s)$ has. Call the partition on S by the same letter R as the partition on F. Then $(S, R, S \ni s \mapsto Y(s) \bullet S)$ is a familoid, and Y is a skif homomorphism from S to F. The meaning of $S \ni s \mapsto Y(s) \bullet S$ as identifier of a transversal-valued function on the partite set (S, R) is that if Q is the function on S going as $Q(s) = Y(s) \bullet S = \{Y(s) \bullet t : t \in S\}$ for each $s \in S$, then (S, R, Q) is a familoid, as, for each $s \in S$, $Y(s) \bullet$ is constant on pieces of R in S, and the constant value on a piece is in that same piece; that proves (S, R, Q) is a familoid. Moreover, denoting the skif products in F and S by juxtaposition, consider $Y(ab)$ and $Y(a)\,Y(b)$. For intuition let's call the elements of F **cats**, the elements of S **seeds**, and the cat Y(s) the **source** of a seed s, and think of the seed span $w \bullet$ for cat w in the action • of F on S as the seeds that joined from the parents of w to make w. The sources of the parent seeds of a seed, seed s, are the parent cats of the source cat of s. The cat $Y(a)\,Y(b)$ has the same sex label as the seed b. So does the cat $Y(ab)$. The cat $Y(a)\,Y(b)$ is a parent of the cat $Y(a)$. The cat $Y(ab)$ is the source of a seed parent of the seed a, so $Y(ab)$ is a parent cat of the source cat of seed a, that is $Y(ab)$ is a parent cat of $Y(a)$, so $Y(ab)$ and $Y(a)\,Y(b)$ are parents of $Y(a)$ of the same sex, hence equal, hence Y is a homomorphism as claimed.

The familoid concept embellishes the concept of partition of a set. Familoids with seeds and familoids with matings* are embellishments on the basic familoid concept.

*That last sentence too was there in that paper, where "familoids with matings" had already been described as people mapping into a mating that maps into transversals of the people sex partition instead of people mapping directly to their parents transversal.

I made up the math idea of familoid to live with what happened to me in a campus counseling clinic. A critique that seemed unlikely to ever be given a chance to engage still needed to be represented, and seeing as I looked to have to keep it to myself for a long time, I decided to pick a math structure that I could stay interested in as math and use in describing aspects of the clinic problem. Familoid was the structure I picked. I have told you some of the math I've done with it in 35 years. I never got anyone to engage with me on the people and abstract interface of human familoid algebra. Never got to say what troubled me about the clinic or what I went there for, but I got some real math done. The clinic problem never was impotence, they made that wrong assumption about what kind of concern about sex I wanted counseling sessions to talk about in, but they didn't tell me much of their reasoning, and I was faced with needing a long time to do a credible, impassive, interesting-to-read writing of what they thought, what happened, what my goal was, what their goal was, and how far the treatment was from deserved or appropriate. My internet feed tells me that the new young adults are interested in mental health care and nonbinary sex systems, so maybe they'll like me for making up familoids in 1989 as a nonbinary sex idea for mental health sake, even though that timing makes me an old guy in 2025.

David Pasino
September 27, 2025 (Tigers clinched a playoff spot today, winnng in Fenway.)