Algorithms for computing A279413 and from this A279414, A186434 and A271908

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1 Introduction

A279413: Triangle read by rows: $T(N, K), N >= K >= 1,$ is the number of isosceles triangles that have a bounding box of size NXK.

A279414: Number of unique isosceles triangles that can be formed from the n^2 points of nXn grid of points.

A186434: Number of isosceles triangles that can be formed from the n^2 points of nXn grid of points (or geoboard).

A271908: $A186434(n)/4$.

The algorithm for A279413 is presented first, and from this A186434, A271908 and A279414 are computed.

2 Notation

$\odot$ The triangle vertex between the same-length sides.

$\oplus$ Alternative positions for some point.

a, b, c The vertices of the triangles.

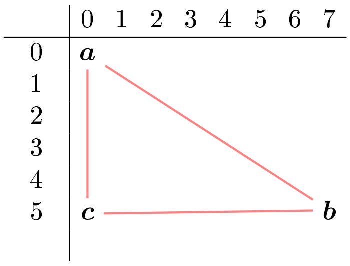
len The length of the two equal sides.

In the following, let $n = N - 1, k = K - 1$.

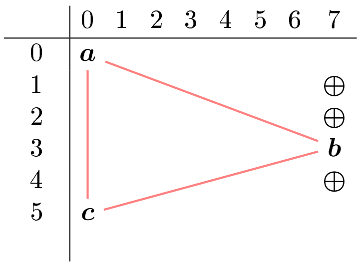
3 Classification of configurations

The configurations of the three points are classified according to which corner points are occupied: A) Three, B) Two adjacent, C) Two opposite, D) One. The illustrations have $\boldsymbol{n = 7, k = 5}$.

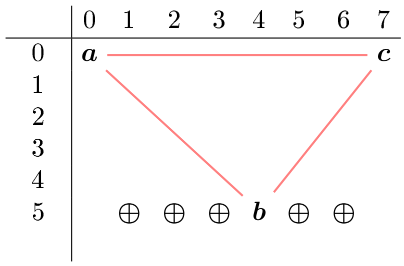
A)



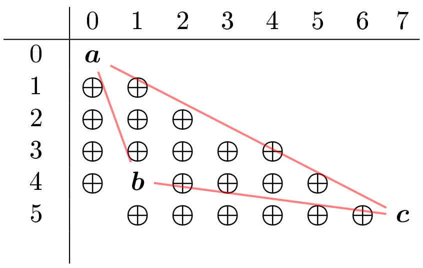
B1)



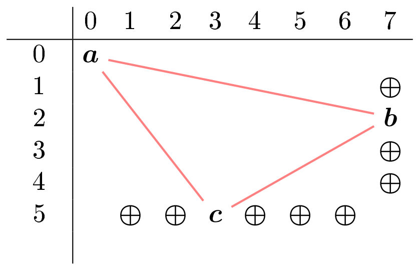
B2)



C)



D)



In A) Only $\boldsymbol{c}$ can be $\odot$, obviously.

In B1) $\boldsymbol{a}$ or $\boldsymbol{b}$ can be $\odot$, but not $\boldsymbol{c}$ because of symmetry (swapping $\boldsymbol{a} \leftrightarrow \boldsymbol{c}$ is the same problem), and $\boldsymbol{b}$ cannot occupy the right corners because then we have configuration A).

In B2) $\boldsymbol{a}$ or $\boldsymbol{b}$ can be $\odot$, but not $\boldsymbol{c}$ because of symmetry, and $\boldsymbol{b}$ cannot occupy the bottom corners because then we have configuration A).

In C) $\boldsymbol{a}$ and $\boldsymbol{b}$ are fixed in opposite corners and $\boldsymbol{b}$ is allowed to take any position within the triangle below the line $\boldsymbol{ac}$. Only b can be $\odot$ and cannot occupy the bottom-left corner because them we have configuration A).

In D) $\boldsymbol{a}$ is fixed in the top left corner, $\boldsymbol{b}$ takes positions on the right side, and $\boldsymbol{c}$ takes positions on the bottom side. Any of $\boldsymbol{a, b, c}$ can be $\odot$ and $\boldsymbol{b, c}$ cannot occupy a corner since then we get configuration A) or B).

3.1 Case A)

Only c can be $\odot$ and solutions are only possible when $k = n$ and then there are 4 solutions taking mirroring into account.

3.2 Case B1)

Letting $b = \odot$ then if k is odd there are 2 solutions (with b to the left and to the right).

Letting $a = \odot$ and setting $b = (r, n)$ we must have
$n^2 = r^2 + n^2$ which is only obtained when $k = n$ and then we must have $r = 0$, which is not allowed, so there are no solutions to this configuration.

3.3 Case B2)

Letting $b = \odot$ then if n is odd there are 2 solutions (with b in top or bottom).

Letting $a = \odot$ and setting $b = (k, s)$ we must have $n^2 = s^2 + k^2$. For $k = n$ we get $s = 0$ which is not allowed.
Otherwise $len = sqrt(n^2 - k^2)$ which is a solution when len is an integer. Each len gives 4 solutions with mirroring.

3.4 Case C)

Let b have coordinates $row, column = r, s$.
Then we must have $r^2 + s^2 = (k - r)^2 + (k - s)^2$, or
$r^2 + s^2 = k^2 + r^2 - 2rk + n^2 + s^2 - 2sn$, or
$2rk + 2sn = k^2 + n^2$,
which is a diophantine equation in r, s.

Solve this equation and keep the points (if any) which are within the bounding box and lie below the line $a - c$.
By mirroring in vertical and horisontal we get 4 times as many solutions.
Note that by construction there is no risk that the third side is equal to the other two.

3.5 Case D)

Place $\odot$ in a.
Then $len = n^2 + b^2 = k^2 + c^2 \neq (n - c)^2 + (k - b)^2$.
Solving for c gives $c^2 = b^2 + (n^2 - k^2)$
When $k = n$ we get $b = c$ so there are $n - 1$ solutions.
Otherwise iterate $b = 1 \ldots k - 1$ and count those for which $b^2 + (n^2 - k^2)$ is a perfect square and $0 < c < n$.

Place $\odot$ in b.
Then $len = n^2 + b^2 = (n - c)^2 + (k - b)^2 \neq k^2 + c^2$.
$n^2 + b^2 = n^2 - 2cn + c^2 + k^2 - 2bk + b^2$
$0 = -2cn + c^2 + k^2 - 2bk \ldots$ (a)
$c^2 - 2cn + k^2 - 2bk = 0$

Solving for c yields $c = \frac{1}{2}(2n + -\sqrt{4n^2 - 4(k^2 - 2bk})$.
$c = n + -\sqrt{n^2 + 2bk - k^2}$.
We require $0 < c < n$
Or, $0 < n + -\sqrt{n^2 + 2bk - k^2} < n$ so we must choose the negative sign.
$-n < -\sqrt{n^2 + 2bk - k^2} < 0$
$n > \sqrt{n^2 + 2bk - k^2} > 0$
$0 < \sqrt{n^2 + 2bk - k^2}) < n$.
For a solution $n^2 + 2bk - k^2$ must be a square, call it $d^2, 0 < d < n$.
So $d^2 = n^2 + 2bk - k^2$, and
$b = \frac{d^2+k^2-n^2}{2k}$
Try all $0 < d < n$ and count those that make $d^2+k^2-n^2$ divisible by $2k$ such that $0 < b < k$.
(and $c = n - d$).

Place ⊙ in c.
Then $s = k^2 + c^2 = (n - c)^2 + (k - b)^2 \neq n^2 + b^2$.
Similar calculations as above yields:
$c = \frac{d^2+n^2-k^2}{2n}$
Try all $0 < d < k$ and count those that makes $d^2+n^2-k^2$ divisible by $2n$ such that $0 < c < n$
(and $b = k - d$).

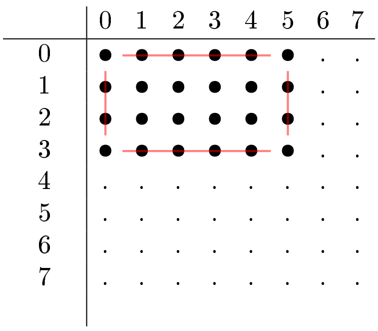
Note that in all the three cases we must check that the third side is not equal len (which never happens for $n, k < 10000$).

By mirroring in vertical and horisontal we get 4 times as many solutions.

4 A186434 and A271908

$A186434$ and $A271908 = A186434/4$ can now be obtained in the following way.

Within a square of size N (here $N = 8$) all rectangles (some are squares) of size $(n, m), m \leq m \leq N$ are placed in all possible positions, illustrated here by $n = 6, m = 4$.

A)


All isosceles triangles within all rectangles are unique. This remains to be proved.

A rectangle can be placed in $N - n + 1$ positions horisontally (here $8 - 6 + 1 = 3$) and in $N - k + 1$ positions vertically (here $8 - 4 + 1 = 5$).

So the total number of positions for the rectangle are $S = (N - n + 1) * (N - k + 1)$ (here $S = 3 * 5 = 15$).

And if, as in the illustration, it is a true rectangle we can mirror it in one of the diagonals, giving twice as many positions $\mathbf{2 * S}$ (here $\mathbf{2 * 15 = 30}$).

Finally, by multiplying by the number of triangles within this rectangle which is given by $\mathbf{A279413(n, k)}$ we obtain the total $\mathbf{2 * S * A279413(n, k)}$ (here $\mathbf{A279413(6, 4) = 8}$ so $\mathbf{2 * 15 * 8 = 240}$).

However, when $\mathbf{n = k}$ the mirroring does not yield a different form so in this case the factor $\mathbf{2}$ must be left out.

Summing over all rectangles/squares that fit into $\mathbf{N * N}$ we obtain:

$\mathbf{A186434(N) = \sum_{n=1}^{N} \sum_{k=1}^{n} \eta * (N - n + 1) * (N - k + 1) * A279413(n, k)}$ where $\boldsymbol{\eta} = \mathbf{1}$ when $\mathbf{k = n}$ and $\boldsymbol{\eta} = \mathbf{2}$ otherwise.

And A279414 are the row sums of A279413:
$\mathbf{A279414(N) = \sum_{k=1}^{N} A279413(n, k)}$.

5 Summary

Implementing this algorithm in $\mathbf{C\#}$ and running it for $\mathbf{N, K = 1..1000}$, the $\mathbf{499, 500}$ terms of A279413 are generated in about one hour.

The 10,000 terms of A186434 took another 14 hours.

Checks have been made against a brute force implementation derived from Nathaniel Johnston's C program in A186434 up to $\mathbf{n = 200}$.