Some Results About Sequence A276710

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Abstract

Using p-adic valuations, membership in OEIS sequence A276710 can be tested with less need for big-integer arithmetic. This can be used to prove a conjecture about the sequence, find a simple description of its complement, and find its natural density.

The sequence [OEIS, sequence A276710 = Composite numbers m such that $\prod_{k=0}^{m}\binom{n}{k}$ is divisible by m^{m-1}] is defined in terms of a divisibility condition among two rapidly growing sequences, namely [OEIS, sequence A001142 = $\prod_{k=1}^{n} k^{2k-1-n}$] and [OEIS, sequence A000169 = Number of labeled rooted trees with n nodes: n^{n-1}]. For example, the fact that 36 occurs (as first entry) in A276710 is a consequence of A001142(36) = 86,647,553,659,528,728,386,865,966, 899,245,661,096,760,113,846,435,205,421,585,685,014,989,365,916,210,134,167, 792,444,618,029,727,631,597,023,317,222,609,765,010,953,506,205,837,870,873, 477,303,408,717,593,918,746,140,769,619,826,402,897,127,023,799,084,181,990, 301,478,751,771,142,735,790,080,000,000,000,000,000,000,000,000,000,000,000 being a multiple of A000169(36) = 2,955,204,414,547,681,244,658,707,659,790, 455,381,671,329,323,051,646,976.

Clearly, one should try to explore the sequence via much smaller numbers — such as p-adic valuations.

1 Definitions and Notation

We write $\mathbb{N}$ for the set of positive integers, $\mathbb{N}_0$ for the set of non-negative integers, $\mathbb{P}$ for the set of primes. In this article, the word "number" and all *variables* written as lower-case latin letters (a, b, c, etc.) refer to non-negative integers only (and specifically p will always refer to a prime). Of course, this does not prevent *expressions* such as $a-b$, $\frac{a}{b}$, or $\sqrt{a}$ from leaving that realm.

Let p be a prime. If n is expressed in base p as

$$n = \sum_{j=0}^{t} a_j p^j$$

with $0 \le a_j < p$, then recall that we define the p-adic valuation

$$v_p(n) := \min\{\, j \in \mathbb{N}_0 : a_j \neq 0 \,\}$$

(and $v_p(0) = \infty$, but this will not occur below) and the p-adic digit sum

$$s_p(n) := \sum_{j=0}^{t} a_j.$$

For a number $n > 1$, let

$$\operatorname{gpf}(n) := \max\{\, p \in \mathbb{P} : p \mid n \,\}$$

denote the greatest prime factor of n.

Specific to our investigation, we abbreviate

$$f_p(n) := v_p\left(\prod_{k=1}^{n-1} \binom{n}{k}\right)$$

and for convenience,

$$\mathbf{A} := \{\, \mathrm{A276710}(n) : n \in \mathbb{N} \,\}.$$

In section 3, we will consider the following sets of low, high, and zero values:

$$\begin{aligned}
\mathbf{L}_p &:= \{\, n \in \mathbb{N} : f_p(n) < (n-1)v_p(n) \,\}, \\
\mathbf{H}_p &:= \{\, n \in \mathbb{N} : f_p(n) > n v_p(n) + \tfrac{1}{p-1} s_p(n) \,\}, \\
\mathbf{Z}_p &:= \{\, n \in \mathbb{N} : f_p(n) = 0 \,\}.
\end{aligned}$$

Note that numbers n with $(n-1)v_p(n) < n \le n v_p(n) + \frac{1}{p-1} s_p(n)$ are in neither of these sets. The main task in that section is to show that such n do not exist.

For subsets $\mathbf{X} \subseteq \mathbb{N}$ the natural density is defined as

$$\varrho(\mathbf{X}) := \lim_{n \to \infty} \frac{\big|\mathbf{X} \cap [1, n]\big|}{n}$$

(if the limit exists). In section 4, we will apply this to the following sets:

$$\begin{aligned}
\mathbf{B}_{r,s}^{(p)} &:= \{\, (c+1)p^r - p^s : 1 \le c < p \,\} \\
\mathbf{B}_{r,s} &:= \bigcup_{p \in \mathbb{P}} \mathbf{B}_{r,s}^{(p)} \\
\mathbf{B}_r &:= \bigcup_{s=1}^{r} \mathbf{B}_{r,s} \\
\mathbf{B}^{(p)} &:= \bigcup_{r=1}^{\infty} \bigcup_{s=1}^{r} \mathbf{B}_{r,s}^{(p)} \\
\mathbf{B} &:= \bigcup_{r=1}^{\infty} \mathbf{B}_r.
\end{aligned}$$

2 Basic Calculations

With the notation introduced in section 1, we have $n \in \mathbf{A}$ if and only if n is composite and

$$f_p(n) \geq (n-1)v_p(n) \tag{1}$$

holds for all primes p (it trivially holds when $p \nmid n$).

By definition,

$$\begin{aligned} f_p(n) &= v_p\left(\prod_{k=0}^{n} \binom{n}{k}\right) \\ &= \sum_{k=0}^{n} v_p\left(\frac{n!}{k!(n-k)!}\right) \\ &= (n+1)v_p(n!) - 2\sum_{k=0}^{n} v_p(k!) \\ &= (n-1)v_p(n!) - 2\sum_{k=0}^{n-1} v_p(k!). \end{aligned} \tag{2}$$

It is well-known that the p-adic valuation of a factorial satisfies the recursion

$$v_p(k!) = \lfloor k/p \rfloor + v_p(\lfloor k/p \rfloor !) \tag{3}$$

and (readily following by induction) the also well-known closed expressions

$$v_p(k!) = \sum_{j=1}^{\infty} \left\lfloor \frac{k}{p^j} \right\rfloor \tag{4}$$

$$= \frac{k - s_p(k)}{p-1}. \tag{5}$$

We see from (3) that for $0 \leq r < p$, the value of $v_p\big((pk+r)!\big) = k + v_p(k!)$ does not depend on r. Therefore, if $n = pm + r$ with $0 \leq r < p$, then

$$\begin{aligned} \sum_{k=0}^{n-1} v_p(k!) &= \sum_{k=0}^{m-1}\sum_{j=0}^{p-1} v_p\big((pk+j)!\big) + \sum_{j=0}^{r-1} v_p\big((pm+j)!\big) \\ &= \sum_{k=0}^{m-1}\sum_{j=0}^{p-1}\big(k + v_p(k!)\big) + \sum_{j=0}^{r-1} v_p(n!) \\ &= p\sum_{k=0}^{m-1}\big(k + v_p(k!)\big) + r v_p(n!) \\ &= p\sum_{k=0}^{m-1} v_p(k!) + \frac{pm(m-1)}{2} + r v_p(n!), \end{aligned}$$

or,

$$\sum_{k=0}^{n-1} v_p(k!) - p\sum_{k=0}^{m-1} v_p(k!) = \frac{pm(m-1)}{2} + rv_p(n!).$$

By using this with (2),

$$\begin{aligned}
f_p(n) - pf_p(m) &= (n-1)v_p(n!) - 2\sum_{k=0}^{n-1} v_p(k!) - p(m-1)v_p(m!) + 2p\sum_{k=0}^{m-1} v_p(k!) \\
&= (n-1)v_p(n!) - p(m-1)v_p(m!) - pm(m-1) - 2rv_p(n!) \\
&= (pm-r-1)v_p(n!) - p(m-1)(m + v_p(m!)) \\
&= (pm-r-1)v_p(n!) - (pm-p)v_p(n!) \\
&= (p-r-1)v_p(n!), \qquad (6)
\end{aligned}$$

which gives us a nice recursion to compute $f_p(n)$. In the special case $r = 0$, we use (5) to turn this into

$$f_p(n) = pf_p(\tfrac{n}{p}) + (p-1)v_p(n!) = pf_p(\tfrac{n}{p}) + n - s_p(n) \qquad \text{if } p \mid n. \qquad (7)$$

Lemma 1. *If $n = \sum_{j=0}^{t} a_j p^j$ with $a_j \in \{0, \ldots, p-1\}$, then*

$$f_p(n) = \sum_{0 \le i \le k < j \le t} (p-1-a_i)a_j p^k.$$

Proof. Note that

$$\left\lfloor \frac{n}{p^j} \right\rfloor = \sum_{i=0}^{t-j} a_{i+j} p^i$$

so that by (4),

$$v_p(n!) = \sum_{j=1}^{t}\sum_{k=0}^{t-j} a_{k+j}p^k = \sum_{0 \le k < j \le t} a_j p^k.$$

With the base case $n = 0$ being trivial, the claim follows by induction using (6):

$$\begin{aligned}
f_p\left(\sum_{j=0}^{t} a_j p^j\right) &= pf_p\left(\sum_{j=0}^{t-1} a_{j+1}p^j\right) + (p-1-a_0)\sum_{0 \le k < j \le t} a_j p^k \\
&= p\sum_{0 \le i \le k < j \le t-1} (p-1-a_{i+1})a_{j+1}p^k + (p-1-a_0)\sum_{0 \le k < j \le t} a_j p^k \\
&= \sum_{0 \le i \le k < j \le t-1} (p-1-a_{i+1})a_{j+1}p^{k+1} + (p-1-a_0)\sum_{0 \le k < j \le t} a_j p^k \\
&= \sum_{1 \le i \le k < j \le t} (p-1-a_i)a_j p^k + (p-1-a_0)\sum_{0 \le k < j \le t} a_j p^k \\
&= \sum_{0 \le i \le k < j \le t} (p-1-a_i)a_j p^k.
\end{aligned}$$

□

3 Strengthening Inequality (1)

Lemma 2. *Let n be a positive integer. If $n = (c+1)p^t - 1$ for some $t \geq 0$ and $1 \leq c < p$, then $f_p(n) = 0$. For any other n, we have*

$$f_p(n) \geq \frac{s_p(n)}{p-1} > 0.$$

Proof. Write n in base p as $n = \sum_{j=0}^{t} a_j p^j$ with $0 \leq a_i < p$ and $a_t \neq 0$. Then $n = (c+1)p^t - 1$ is equivalent to: $a_i = p-1$ for all $i < t$, and $a_t = c$. For such n, all summands in lemma 1 are zero, hence $f_p(n) = 0$.

For any other n, some non-leading p-ary digit is not $p-1$, so let $r = \min\{\, i \in \mathbb{N}_0 : a_i < p-1 \,\}$. Then $r < t$ and by taking only the summands with $i = k = r$ in lemma 1, we obtain the (somewhat generous) lower estimate

$$\begin{aligned} f_p(n) &\geq \sum_{j=r+1}^{t} (p-1-a_r) a_j p^r \\ &\geq p^r \sum_{j=r+1}^{t} a_j \\ &\geq \sum_{j=r+1}^{t} a_j + (p^r - 1) \\ &\geq \sum_{j=r+1}^{t} a_j + (p-1)r \\ &= s_p(n) - a_r. \end{aligned}$$

If $p = 2$, then $a_r = 0$ and $f_p(n) \geq s_p(n)$ as desired. And if $p > 2$, then

$$\frac{f_p(n)}{s_p(n)} \geq \frac{s_p(n) - a_r}{s_p(n)} = 1 - \frac{1}{1 + \frac{s_p(n)-a_r}{a_r}} \geq 1 - \frac{1}{1 + \frac{1}{p-2}} = \frac{1}{p-1}.$$

□

Lemma 3. $p\mathbf{L}_p \subseteq \mathbf{L}_p$ *and* $p\mathbf{H}_p \subseteq \mathbf{H}_p$.

Proof. Let $n = pm$, so $v_p(n) = v_p(m) + 1$ and $s_p(n) = s_p(m)$. If $m \in \mathbf{L}_p$, then from (7), we find

$$\begin{aligned} f_p(n) &= p f_p(m) + n - s_p(n) \\ &< p(m-1)v_p(m) + n - s_p(n) \\ &= n v_p(m) - p v_p(m) + n - s_p(n) \\ &= n v_p(n) - p v_p(m) - s_p(n) \\ &\leq n v_p(n) - v_p(m) - 1 \\ &= (n-1) v_p(n). \end{aligned}$$

Similarly, if $m \in \mathbf{H}_p$, then

$$\begin{aligned} f_p(n) &= pf_p(m) + n - s_p(n) \\ &> p\big(mv_p(m) + \tfrac{1}{p-1}s_p(m)\big) + n - s_p(n) \\ &= pmv_p(m) + \tfrac{p}{p-1}s_p(n) + n - s_p(n) \\ &= nv_p(n) + \tfrac{1}{p-1}s_p(n). \end{aligned}$$

□

Lemma 4. $\mathbf{Z}_p = \{\, (c+1)p^t - 1 : t \in \mathbb{N}_0, 1 \le c < p \,\}$.

Proof. Immediate from lemma 2 and the definition of $\mathbf{Z}_p$. □

Lemma 5. $\mathbf{Z}_p = (p\mathbf{Z}_p + p - 1) \cup \{1, \dots, p-1\}$. *In particular,* $\mathbf{Z}_p \cap p\mathbb{N} = \emptyset$.

Proof. Suppose $n \in \mathbf{Z}_p$, so by lemma 4, $n = (c+1)p^t - 1$ with $t \ge 0$ and $1 \le c < p$. If $t > 0$, then $n = p \cdot \big((c+1)p^{t-1} - 1\big) + p - 1 \in p\mathbf{Z}_p + p - 1$. If $t = 0$, then $n = c \in \{1, \dots, p-1\}$. Conversely, if $n = (c+1)p^t - 1$ as in lemma 4, then $pn + p - 1 = cp^{t+1} - 1 \in \mathbf{Z}_p$, and if $1 \le n \le p-1$, then $n = (n+1)p^0 - 1 \in \mathbf{Z}_p$, thus showing the first claim. The second claim follows from the first as every $n \in \mathbf{Z}_p$ is $\equiv -1 \pmod p$ or $< p$. □

Lemma 6. *For fixed p, the sets* $\mathbf{L}_p$, $\mathbf{H}_p$, $\mathbf{Z}_p$ *are disjoint.*

Proof. We have $\mathbf{L}_p \cap \mathbf{H}_p = \emptyset$ and $\mathbf{H}_p \cap \mathbf{Z}_p = \emptyset$ immediately from the defining predicates. If $n \in \mathbf{Z}_p$ then $p \nmid n$ by lemma 5, hence $v_p(n) = 0$ and we cannot have $f_p(n) < (n-1)v_p(n)$. Thus also $\mathbf{L}_p \cap \mathbf{Z}_p = \emptyset$. □

Consider the finite state automaton $\mathcal{M}_p$ depicted in fig. 1, where an arrow with label $\star$ is understood to stand for several arrows: one arrow with label r for each $r \in \{0, \dots, p-1\}$ for which there is not already an arrow with explicit label r and same source node.

Lemma 7. *In the finite state automaton* $\mathcal{M}_p$, *consider an arrow* $\mathbf{X} \xrightarrow{r} \mathbf{Y}$ *with label r from a node with label* $\mathbf{X}$ *to a node with label* $\mathbf{Y}$. *Then for every* $m \in \mathbf{X}$, *we have* $pm + r \in \mathbf{Y}$.

Proof. Let $n = pm + r$ with $0 \le r \le p-1$. We distinguish cases guided by the nodes and arrows in fig. 1.

- $m = 0$: The cases $r = 0$ and $r = 1$ are clear. For $r \ge 2$, we have $n \in \mathbf{Z}_p$ by lemma 5.
- $m = 1$: The case $r = 0$ is clear. As $1 \in \mathbf{Z}_p$, lemma 5 implies that $n \in \mathbf{Z}_p$ if $r = p-1$ and $n \notin \mathbf{Z}_p$ for $1 \le r < p-1$. In the latter case, $f_p(n) > 0$ by lemma 2, but $v_p(n) = 0$, so $n \in \mathbf{H}_p$.

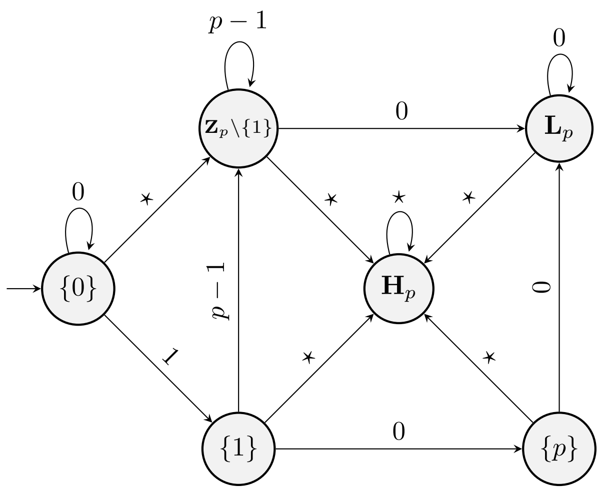

Figure 1: Finite state automaton $\mathcal{M}_p$ used in the proof of lemma 7.

- $m = p$: By lemma 5, $n \notin \mathbf{Z}_p$. For every $r > 0$, we have $v_p(n) = 0$ and so by lemma 2 must have $n \in \mathbf{H}_p$. If $r = 0$, then by (7),

$$\begin{aligned} f_p(n) &= f_p(p^2) = pf_p(p) + p^2 - s_p(p^2) \\ &= p(p-1) + p^2 - 1 = 2p^2 - p - 1 < (p^2-1)v_p(p^2) \end{aligned}$$

  and so $n \in \mathbf{L}_p$.

- $m \in \mathbf{X}$ where $\mathbf{X} = \mathbf{L}_p$ or $\mathbf{X} = \mathbf{H}_p$: If $r = 0$, then $n \in \mathbf{X}$ by lemma 3. For other r, note that $m \notin \mathbf{Z}_p$ by lemma 6, hence $n \in \mathbf{H}_p$ by lemma 5.

- $m \in \mathbf{Z}_p \setminus \{1\}$: If $r = p-1$, then $n \in \mathbf{Z}_p$ by lemma 5. For any other non-zero r, we have $n \in \mathbf{H}_p$ by lemma 5 and lemma 2. By lemma 4, $v_p(m) = 0$ and $m = (c+1)p^t - 1$ where either $s_p(m) \geq p-1$, or $t = 0$ and $s_p(m) = m > 1$. At any rate, $s_p(m) > 1$. Thus for $r = 0$, we have $v_p(n) = 1$ and $s_p(n) > 1$, so from (7), we obtain $f_p(n) = n - s_p(n) < n-1$ and thus $n \in \mathbf{L}_p$.

□

Theorem 1. *For every prime p, we have $\mathbb{N} = \mathbf{L}_p \sqcup \mathbf{H}_p \sqcup \mathbf{Z}_p \sqcup \{p\}$.*

Proof. Disjointness follows from lemma 6 and $f_p(p) = (p-1) = (p-1)v_p(p)$. If we feed the base p expansion of $n \in \mathbb{N}$ from highest to lowest place as input into the finite state automaton $\mathcal{M}_p$, then by induction using lemma 7, we end up in a state labeled with a set having n as element. As $1 \in \mathbf{Z}_p$, we see that n is in $\mathbf{L}_p \cup \mathbf{H}_p \cup \mathbf{Z}_p \cup \{p\}$. □

Corollary 1. *For $n \in \mathbb{N}$, we have $n \in \mathbf{A}$ iff $n > 1$ and $n \in \mathbf{H}_p$ for all prime divisors p of n.*

Proof. Assume $n \in \mathbf{A}$. Then $n > 1$ and (1) holds for every prime p. For those with $v_p(n) > 0$, this implies $n \notin \mathbf{L}_p$ and $n \notin \mathbf{Z}_p$. As $n \neq p$, theorem 1 implies $n \in \mathbf{H}_p$.

Conversely, $n \in \mathbf{H}_p$ for all primes $p \mid n$ clearly implies (1) for these (and trivially for $p \nmid n$). As $p \notin \mathbf{H}_p$ and $n \neq 1$, n must be composite. Hence $n \in \mathbf{A}$. $\square$

Corollary 2.

$$\mathbb{N} = \{1\} \sqcup \mathbf{A} \sqcup \bigcup_{p \in \mathbb{P}} (\mathbf{L}_p \cup \{p\}).$$

Proof. By the very definitions and (1), $\mathbf{A}$ contains only composite numbers and is disjoint from every $\mathbf{L}_p$. On the other hand, if $n \notin \mathbf{A}$, then either $n = 1$ or according to corollary 1, there is a prime p with $p \mid n$ and $n \notin \mathbf{H}_p$. By lemma 5, also $n \notin \mathbf{Z}_p$, hence by theorem 1, we have either $n \in \mathbf{L}_p$ or $n = p$. $\square$

Corollary 3. *If $n \in \mathbf{A}$, then*

$$n^n \mid \prod_{k=0}^{n} \binom{n}{k}.$$

Proof. Suppose $n \in \mathbf{A}$. Then by corollary 1, $f_p(n) \geq n v_p(n)$ for all prime divisors p of n (and trivially also for those p not dividing n), which is equivalent to the claim. $\square$

4 Finding the Density

As the corresponding out-arrows of nodes $\{p\}$ and $\mathbf{L}_p$ in fig. 1 have the same targets, we can merge these states. After that, we can likewise merge states $\{1\}$ and $\mathbf{Z}_p \setminus \{1\}$. The result $\mathcal{M}'_p$ is shown in fig. 2 and reflects the result of corollary 3 insofar as we are left with one state for $f_p(n) > n v_p(n)$, one for $f_p(n) < n v_p(n)$, and one for $f_p(n) = n v_p(n) = 0$. The node $\mathbf{L}_p \cup \{p\}$ is marked as accepting state and it corresponds to all inputs n with $f_p(n) < n v_p(n)$ (and in particular, $p \mid n$). By corollary 1, input n (or more precisely, its base p expansion) is accepted by $\mathcal{M}'_p$ if p "witnesses" that $n \notin \mathbf{A}$. We read off the regular expression

$$[\,1 \mid \cdots \mid p-1\,]\,(p-1)^*\,0\,0^*$$

for the inputs accepted by $\mathcal{M}'_p$. Equivalently, n is accepted iff $n \in \mathbf{B}^{(p)}$. We conclude $\mathbf{B}^{(p)} = \mathbf{L}_p \cup \{p\}$.

Note that

$$\bigcup_{p \in \mathbb{P}} \mathbf{B}^{(p)} = \bigcup_{p \in \mathbb{P}} \bigcup_{r=1}^{\infty} \bigcup_{s=1}^{r} \mathbf{B}^{(p)}_{r,s} = \bigcup_{r=1}^{\infty} \bigcup_{s=1}^{r} \bigcup_{p \in \mathbb{P}} \mathbf{B}^{(p)}_{r,s} = \bigcup_{r=1}^{\infty} \bigcup_{s=1}^{r} \mathbf{B}_{r,s} = \bigcup_{r=1}^{\infty} \mathbf{B}_r = \mathbf{B}.$$

Hence we can rewrite corollary 2 as

$$\mathbb{N} = \{1\} \sqcup \mathbf{A} \sqcup \mathbf{B}. \tag{8}$$

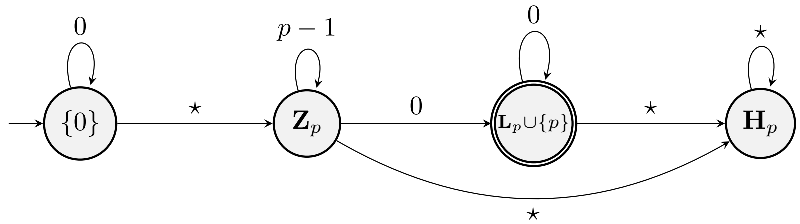

Figure 2: Finite state automaton $\mathcal{M}'_p$ derived from $\mathcal{M}_p$.

Lemma 8. $\varrho(\mathbf{B}_1) = \ln 2$.

Proof. Note that

$$\mathbf{B}_1 = \{\, cp : p \in \mathbb{P}, 1 \le c < p \,\} = \left\{\, n \in \mathbb{N} : n > 1, \operatorname{gpf}(n) > \sqrt{n} \,\right\} \tag{9}$$

so that $\mathbf{B}_1$ is the complement of $\{\, \text{A048098}(n) : n \in \mathbb{N} \,\}$. The latter is known to have density $1 - \ln 2$, cf. [OEIS, sequence A048098 = Numbers k that are $\sqrt{k}$-smooth: if $p \mid k$ then $p^2 \le k$ when p is prime] and [Ram, p. 1125]. $\square$

Lemma 9. *If $r \ge 2$ and $1 \le s \le r$, then $\varrho(\mathbf{B}_{r,s}) = 0$.*

Proof. Given $r \ge 2$ and $1 \le s \le r$, define the map

$$\begin{aligned} g \colon \mathbf{B}_1 &\to \mathbb{N} \\ n &\mapsto n \operatorname{gpf}(n)^{r-1} + \operatorname{gpf}(n)^r - \operatorname{gpf}(n)^s \end{aligned}$$

In other words, if $1 \le c < p \in \mathbb{P}$, then $g(cp) = (c+1)p^r - p^s$, so the image of g is clearly $\mathbf{B}_{r,s}$. By (9), we have

$$g(n) \ge n \operatorname{gpf}(n)^{r-1} > n^{\frac{r+1}{2}} \ge n^{\frac{3}{2}}.$$

From this,

$$\frac{\left|\mathbf{B}_{r,s} \cap [1,n]\right|}{n} \le \frac{\left|\mathbf{B}_1 \cap [1, n^{\frac{2}{3}}]\right|}{n} \le n^{-\frac{1}{3}}$$

and so $\varrho(\mathbf{B}_{r,s}) = 0$. $\square$

Lemma 10. $\varrho\left(\bigcup_{r=3}^{\infty} \mathbf{B}_r\right) = 0$.

Proof. By the Prime Number Theorem (or even an extremely weak form of it), there exists a number h such that the prime counting function satisfies

$$\pi(x) < \frac{hx}{\ln x}$$

for all $x \ge 2$.

Let $r \geq 3$ and $1 \leq s \leq r$. We want to estimate $\left|\mathbf{B}_{r,s} \cap [1,n]\right|$ for $n \gg 0$. As $(c+1)p^r - p^s \geq p^r \geq 2^r$, we need only consider $r \leq \log_2 n$, and for such r, we need only consider primes $p \leq \sqrt[r]{n}$. Thus there are

$$\pi\left(\sqrt[r]{n}\right) < \frac{h\sqrt[r]{n}}{\ln \sqrt[r]{n}} = \frac{hr\sqrt[r]{n}}{\ln n}$$

possible choices for p, and then $p \leq \sqrt[r]{n}$ choices for c to form $(c+1)p^r - p^s$. We conclude

$$\left|\mathbf{B}_{r,s} \cap [1,n]\right| < \frac{hr\sqrt[r]{n}}{\ln n} \cdot \sqrt[r]{n} = \frac{hrn^{\frac{2}{r}}}{\ln n} \leq \frac{hrn^{\frac{2}{3}}}{\ln n}.$$

By summing over $1 \leq s \leq r$,

$$\left|\mathbf{B}_r \cap [1,n]\right| < \frac{hr^2 n^{\frac{2}{3}}}{\ln n}.$$

By summing over $3 \leq r \leq \log_2 n$,

$$\left|\bigcup_{r=3}^{\infty} \mathbf{B}_r \cap [1,n]\right| < \frac{hn^{\frac{2}{3}}}{\ln n} \cdot \sum_{r=1}^{\lfloor \log_2 n \rfloor} r^2 = O(n^{\frac{2}{3}} \ln^2 n)$$

and the claim follows. □

Theorem 2. *$\varrho(\mathbf{B}) = \ln 2$ and $\varrho(\mathbf{A}) = 1 - \ln 2$.*

Proof. As

$$\mathbf{B} = \mathbf{B}_1 \cup \mathbf{B}_{2,1} \cup \mathbf{B}_{2,2} \cup \bigcup_{r=3}^{\infty} \mathbf{B}_r,$$

the first claim follows from lemma 8, lemma 9, and lemma 10. The second claim follows form (8). □

References

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