q-Trigonometry: Some Prefatory Afterthoughts

When I wrote this paper in 1981, I got so diverted chasing identities that I forgot why I q-generalized sin, cos, and π in the first place. I now recall it was partly to provide q-factorial (or q-Γ) with reflection and half unit (π) formulas, so as to sharpen or even trivialize the analogies between hypergeometric identities and their q-generalizations, at least in those cases where the (ordinary) factorials simplified to trig functions and πs. Thus, for example, we could rewrite Ramanujan's identity

$$\sum_{k\geq 0} \frac{(q^a, q^{1-a}; q)_k}{(q;q)_{2k}} q^{k^2} = \frac{(q^{a+1}, q^{2-a}; q^3)_\infty}{(q, q^2; q^3)_\infty}$$

(from that darned "Lost" "Notebook") as

$$\text{(Lost)} \qquad \sum_{k\geq 0} \frac{(a+k-1)!_q}{(a-1)!_q} \frac{(k-a)!_q}{(-a)!_q} \frac{q^{k^2}}{(2k)!_q} = \frac{\frac{-2}{3}!_{q^3}}{\frac{a-2}{3}!_{q^3}} \frac{\frac{-1}{3}!_{q^3}}{\frac{-a-1}{3}!_{q^3}} = \frac{\sin_{q^{3/2}} \pi \frac{a+1}{3}}{\sin_{q^{3/2}} \frac{\pi}{3}} q^{(1-a)a/6},$$

where

$$(q^b;q)_c = (1-q)^c \frac{(b+c-1)!_q}{(b-1)!_q} \qquad \text{and} \qquad b!_q = (1-q)^{-b} \frac{(q;q)_\infty}{(q^{b+1};q)_\infty}.$$

Then you can take the $q \to 1$ limit of (Lost) merely by "erasing the qs". What's more, the q-trigonometric form elucidates the asymptosy for large a. Or you could even write

$$\sum_{k\geq 0} \frac{(a+k-1)!_q (k-a)!_q}{(2k)!_q} q^{k^2} = \frac{\pi_{\sqrt{q}}}{\sin_{\sqrt{q}} \pi a} \frac{\sin_{q^{3/2}} \pi \frac{a+1}{3}}{\sin_{q^{3/2}} \frac{\pi}{3}} q^{(a-1)a/3}.$$

Similarly, if we rewrite

$$\text{(Poch)} \qquad \sum_{n\geq 0} \frac{1-q^{12n+1}}{1-q^2} \frac{(q^{-1}, q^3; q^2)_{2n}}{(q^4;q^2)_{4n}} q^{4n^2} = \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty} \frac{(q^{10};q^{10})_\infty}{(q^5;q^{10})_\infty},$$

(which, hope yet glimmers, Ramanujan may have overlooked) as

$$\sum_{n\ge 0}\frac{1-q^{12n+1}}{1-q^2}\frac{(2n-\frac{3}{2})!_{q^2}(2n+\frac{1}{2})!_{q^2}}{(4n+1)!_{q^2}}q^{4n^2}=-q^{1/4}\sqrt{\pi_q\pi_{q^5}\frac{1-q^2}{1-q^{10}}},$$

then we can safely "erase the qs", leaving only $-\pi/\sqrt{5}$ on the right. This is a lot less tricky than taking $q\to 1$ on the right of (Poch):

$$\lim_{q\to 1}\prod_{n\ge 1}\frac{1-q^{2n-1}}{1-q^{2n}}\frac{1-q^{10n}}{1-q^{10n-5}}=\lim_{q\to 1}\prod_{n\ge 0}\frac{1-q^{10n+1}}{1-q^{10n+2}}\frac{1-q^{10n+3}}{1-q^{10n+4}}\frac{1-q^{10n+7}}{1-q^{10n+6}}\frac{1-q^{10n+9}}{1-q^{10n+8}}$$

$$=\prod_{n\ge 0}\frac{10n+1}{10n+2}\frac{10n+3}{10n+4}\frac{10n+7}{10n+6}\frac{10n+9}{10n+8}=\frac{\sin\frac{3\pi}{10}}{2\sin^2\frac{2\pi}{5}}=\frac{1}{\sqrt{5}}.$$

But it was only as I sat down to write this that I recalled a much stronger motivation for $\sin_q$: it q-generalizes a valuable trick for guessing the extension to continuous x and y of $\prod_{n=x}^{y}f(n)$, where f is a rational function of n (or q^n).

For example, one might guess that

$$(+)\qquad \prod_{n=x}^{y}(n+a)=\frac{(y+a)!}{(x+a-1)!}$$

and

$$(-)\qquad \prod_{n=x}^{y}(-n-a)=\frac{(-x-a)!}{(-y-a-1)!}=\frac{(y+a)!}{(x+a-1)!}\frac{\sin\pi(y+a)}{\sin\pi(x+a-1)}$$

Now from a path-invariant matrix grid, I got

$${}_6F_5\left[\begin{matrix}a, & a-b+1, & a+b-1, & \frac{3-a-b}{2}, & \frac{5a-b}{2}, & a-\frac{b}{5}+1\\ & 2a, & a-\frac{b}{2}+\frac{3}{4}, & a-\frac{b}{2}+\frac{5}{4}, & a+b+\frac{1}{2}, & a-\frac{b}{5}\end{matrix}\middle| -\frac{1}{4}\right]$$

(Special)

$$=\prod_{n=0}^{-\frac{a+b+1}{2}}\frac{(n-a+\frac{b}{2}-\frac{1}{4})(n-a+\frac{b}{2}+\frac{1}{4})(n+\frac{a+b}{2})^2}{(n+\frac{a+b}{2}-\frac{1}{4})(n+\frac{a+b}{2}+\frac{1}{4})(n-\frac{a-b}{2})(n-\frac{5a-b}{2})},$$

for integer $\frac{a+b+1}{2}$. I couldn't do the contour limit for the general case, so I applied equation (+) to the product and got

$$\text{(Bogus)}, \qquad {}_6F_5\left[-\frac{1}{4}\right] \stackrel{?}{=} \sqrt{\pi}\frac{(\frac{b-5a}{2}-1)!}{(\frac{a+b}{2}-1)!^2}\frac{(\frac{b-a}{2}-1)!}{(-a-\frac{1}{2})!}\frac{(a+b-\frac{3}{2})!}{(b-2a-\frac{3}{2})!}$$

which, empirically, fails in the noninteger case. But suppose, in the product, an even number of factors of the form $n+x$ are replaced by $-n-x$, and for these negated factors, we use equation $(-)$ instead of $(+)$. We try multiplying by each of the $\binom{6}{0}+\binom{7}{0}+\binom{6}{2}+\binom{7}{2}+\binom{6}{4}+\cdots = 2^5+2^6 = 96$ distinct sub(multi)sets of even cardinality of factors of the form

$$\frac{\sin\pi(-x-\frac{a+b+1}{2})}{\sin\pi(-x-1)},$$

(or its reciprocal), for x a root (or, respectively, a pole) of the "$\Pi$and" in (Special). Then, sure enough,

$$\text{(General)} \qquad {}_6F_5\left[\begin{matrix} a, & a-b+1, & a+b-1, & \frac{3-a-b}{2}, & \frac{5a-b}{2}, & a-\frac{b}{5}+1 \\ & 2a, & a-\frac{b}{2}+\frac{3}{4}, & a-\frac{b}{2}+\frac{5}{4}, & a+b+\frac{1}{2}, & a-\frac{b}{5}\end{matrix}\;\middle|\; -\frac{1}{4}\right]$$

$$= \sqrt{\pi}\frac{(a-\frac{1}{2})!}{(\frac{a+b}{2}-1)!^2}\frac{(2a-b+\frac{1}{2})!}{(\frac{5a-b}{2})!}\frac{(a+b-\frac{3}{2})!}{(\frac{a-b}{2})!},$$

withstands numerical testing. (Because there was an even number of negations, no explicit sin appears in this result, but you can compare with (Bogus) to see which factorials the sins flipped, and hence, which factors were negated.)

But the grid matrices implying (Special) have q-generalizations. To perform the q-analogous experiments, we need a period 2, amplitude 1 function built from reflected q-factorials, that is, a $\sin_q$. Then we can guess

$$(+_q) \qquad \prod_{n=x}^{y}(1-q^{n+a}) = (q^{x+a};q)_{y-x+1} = \frac{(y+a)!_q}{(x+a-1)!_q}(1-q)^{y-x+1}$$

and

$$\prod_{n=x}^{y} -(1-q^{n+a}) = \prod_{n=x}^{y} q^{n+a}(1-q^{-n-a}) = q^{(y-x+1)(\frac{x+y}{2}+a)}(q^{-y-a};q)_{y-x+1}$$

$$(-_q) \qquad = q^{(y-x+1)(\frac{x+y}{2}+a)} \frac{(-x-a)!_q}{(-y-a-1)!_q}(1-q)^{y-x+1}$$

$$= \frac{(y+a)!_q}{(x+a-1)!_q} \frac{\sin_{\sqrt{q}} \pi(y+a)}{\sin_{\sqrt{q}} \pi(x+a-1)}(1-q)^{y-x+1}.$$

I seem to remember deciding to define $\sin_q$ in terms of $!_{q^2}$ instead of simply $!_q$ to cosmetize the definition via ϑ functions, but, in light of the resulting profusion of square rooted subscripts, my resolve is weakening.

At any rate, by the above means, at this very writing, I find

$$\sum_{k\geq 0} \frac{B-A^5q^{5k}}{B-A^5} \frac{(AB/q;q)_k(A^5/B,q^3/AB;q^2)_k(A^3q^3/B^3;q^3)_k(A^6;q^6)_k}{(q^3,A^6;q^3)_k(A^2B^2/q;q^2)_k(A^4q^3/B^2,A^4q^5/B^2;q^4)_k}(-AB)^k q^{(k-1)k/2}$$

$$(q\text{-General}) \qquad = \frac{(AB,A^5q^2/B;q^2)_\infty}{(A^2B^2/q,A^4q^3/B^2;q^2)_\infty} \frac{(A^3B^3,A^3q^6/B^3;q^6)_\infty}{(q^3,A^6q^3;q^6)_\infty}$$

or, with $A := q^a, B := q^b$

$$= \frac{q^{-3/8}\sqrt{\pi_{q^3}}}{(\frac{a+b}{2}-1)!_{q^6}} \frac{(a-\frac{1}{2})!_{q^6}}{(\frac{a+b}{2}-1)!_{q^2}} \frac{(2a-b+\frac{1}{2})!_{q^2}}{(\frac{5a-b}{2})!_{q^2}} \frac{(a+b-\frac{3}{2})!_{q^2}}{(\frac{a-b}{2})!_{q^6}},$$

the q-generalization of (General). Had there been an explicit minus (*i.e.* an odd number of minuses) in the argument of the finite product, at least one explicit $\sin_q$ would have appeared in the final result.

While (q-General) is just as unproven as (General), it is subject to a much more satisfying test—mechanical Taylor expansion about $q = 0$ with a and b left arbitrary symbols.

In (q-General) we didn't really have to search over 96 $\sin_q$ quotients, since we already knew from (General) which factorials to flip. But this will not always be so, since, when $q \to 1$, whole factors can cancel, vanish, or blow up, leaving the q-less identity inutile or even nonexistent.

But even with the benefit of a meaningful "q-specialization", there is reason to try the $\sin_q$s first: the extra structure imposed by the the powers of q and $1-q$ in $(+_q)$ and $(-_q)$ can substantially narrow the (multi)set of candidate flipands. That is, if we conjecture that the flipping of a subset should introduce no net power of $1-q$, then the only flips can be of numerator–denominator pairs, and these must have equal powers of q. There are only 25 such pairings in the (suppressed for brevity) "$\Pi$and" of (q-Special). There would have been only 15, had not the squared factor in (Special) separated into two different q powers, thereby upgrading the multiset to a true set. Best is probably to search in 1-land when possible, but guided by structural hints from q-land.

On the other hand, a q-land search might fail outright, while a 1-land search succeeds, given the possibility of whole *additive* expressions materializing when $|q| < 1$, as with the q-Whipple and q-Watson theorems.

Last, but nevertheleast, is the "reflected q-Stirling's formula", *i.e.* the expansion of $(Z;q)_\infty$ for large Z. The traditional q-Stirling's formula expands $(q^z;q)_\infty$ for large z, which means small Z:

$$(q^{z+y};q)_\infty = \frac{1}{\sum_{k\ge0}\frac{q^{(z+y)k}}{(q;q)_k}} = \sum_{k\ge0}\frac{q^{(z+y-1)k}}{(q^{-1};q^{-1})_k}$$
$$= \exp\sum_{n\ge1}\frac{q^{(z+y)n}}{n(q^n-1)} \asymp_z \exp\sum_{k\ge0} B_k(y)\,\mathrm{Li}_{2-k}(q^z)\frac{(\ln q)^{k-1}}{k!},$$

where the final step follows immediately (without appeal to Euler–Maclaurin summation) from the Bernoulli polynomial generating function,

$$\frac{te^{yt}}{e^t-1} = \sum_{k\ge0} B_k(y)\frac{t^k}{k!},$$

using $e^t = q^n$ and

$$\mathrm{Li}_k(Z) := \sum_{n\ge1}\frac{Z^n}{n^k},$$

which is a rational function of Z for $k \le 0$.

To reflect, we need merely change y to $1-y$ and toss in the appropriate $\sin_{\sqrt{q}}$:

$$(q^{y-z};q)_\infty = \sin_{\sqrt{q}}\pi(y-z)\ \frac{(\sqrt{q};q)_\infty^2}{q^{(y-z-1/2)^2/2}}\sum_{k\ge0}\frac{q^{(z+1-y)k}}{(q;q)_k},$$

which outright converges for large z, (or as a formal power series in q), unless you need the option to translate by y, using the exponential–Bernoulli form.

Thus we elucidate the scale and placement of the poles and swoops of $!_q$, and simplify the taking of limits where $z \to \infty$ by integer steps.

For complex z, the $\sin_q$ and $\cos_q$ addition and subtraction formulas may prove useful, with $x = z$ and $y = \overline{z}$.

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