Notes on A273596

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NAME: The number of slim rectangular diagrams of length $n \geq 2$.

The sequence begins $1, 3, 9, 32, 139, 729, 4515, 32336, 263205, ...$ with an offset of 2.

Formula for the sequence terms

$$a(n) \quad = \sum_{1 \leq r,s;r+s \leq n} (n-r-s)! \binom{n-r-1}{s-1} \binom{n-s-1}{r-1}. \qquad (1)$$

Our aim is to find a simpler formula for $a(n)$ and then using this to produce a recurrence for the sequence.

1. A recurrence equation

First set $r+s=t$ in (1) to put the double sum in the form

$$\begin{aligned} a(n) &= \sum_{t=2}^{n} \sum_{r=1}^{t-1} (n-t)! \binom{n-r-1}{t-r-1} \binom{n-t+r-1}{r-1} \\ &= \sum_{t=2}^{n} s(n,t), \end{aligned} \qquad (2)$$

where $s(n,t)$ denotes the inner sum

$$s(n,t) \quad = \quad \sum_{r=1}^{t-1} (n-t)! \binom{n-r-1}{t-r-1} \binom{n-t+r-1}{r-1} \qquad (3)$$

defined for values $n \geq t$, with the initial value $s(t,t) = t-1$.

We find a recurrence for this sum using Maple. The Maple code

```
with(sumtools):
sumrecursion((n-t)!*binomial(n-r-1,t-r-1)binomial(n-t+r-1,r-1), r, s(n));
```

returns the recurrence equation

$$s(n,t) \quad = \quad \frac{(2n-1-t)(2n-2-t)}{2(2n+1-2t)} s(n-1,t). \qquad (4)$$

One easily verifies that

$$s(n,t) \quad := \quad \frac{2(n+1-t)!(2n-1-t)!}{(2n+2-2t)!(t-2)!} \tag{5}$$

satisfies the recurrence (4) together with the initial condition $s(t,t) = t-1$.

Thus (2) now becomes a representation for the sequence terms as a single sum:

$$a(n) \quad = \quad 2\sum_{t=2}^{n} \frac{(n+1-t)!(2n-1-t)!}{(2n+2-2t)!(t-2)!}, \quad n \geq 2.$$

Changing the dummy variable t to $n-t$ gives

$$a(n) \quad = \quad 2\sum_{t=0}^{n-2} \frac{(t+1)!(n+t-1)!}{(2t+2)!(n-t-2)!}, \quad n \geq 2$$

or equivalently,

$$a(n) \quad = \quad \sum_{t=0}^{n-2} t!\binom{n+t-1}{2t+1}, \quad n \geq 2. \tag{6}$$

The first few summands in (6) are shown in Table 1 below. Since the row sums of this table give the number of slim rectangular diagrams of length $n+2$ (equivalently, the number of rectangular permutations in the symmetric group S_{n+2}) one wonders if the table entries have some combinatorial significance, perhaps as some statistic on rectangular permutations.

	$t=0$	1	2	3	4	5
$n=0$	1					
1	2	1				
2	3	4	2			
3	4	10	12	6		
4	5	20	42	48	24	
5	6	35	112	216	240	120

Table 1: $t!\binom{n+t+1}{2t+1}, 0 \leq t \leq 5, 0 \leq n \leq 5$.

Next we use the ZeilbergerRecurrence command in Maple's SumTools package to find a recurrence for $a(n)$ given by the sum in (6) over the range of summation from $t=0$ to $t=n-2$. The Maple code

```
with(SumTools[Hypergeometric]):
T := factorial(t)*binomial(n+t-1, 2*t+1):
ZeilbergerRecurrence(T, n, t, a, 0 .. n-2);
```

returns the inhomogeneous recurrence equation

$$a(n+2) \quad = \quad na(n+1)+a(n)+2 \tag{7}$$

satisfied by the number of rectangular permutations in S_n. The simplicity of this recurrence suggests it may have a combinatorial proof. The recurrence provides a method for rapidly calculating the terms of the sequence.

2. The ordinary generating function for the sequence as a white diamond product of power series

In [1] we introduced the white diamond product of a pair of formal power series $A(x), B(x) \in \mathbb{C}[[x]]$, denoted symbolically by $A(x) \lozenge B(x)$. The white diamond product is a commutative and associative $\mathbb{C}$-bilinear multiplication of power series. The white diamond product of monomial polynomials is given by the formula [1, Proposition 1]

$$x^m \lozenge x^n \quad = \quad \sum_{k=0}^{m} \frac{m!n!}{(n+k)!}\binom{n+k}{m}\binom{m}{k}x^{n+k}. \tag{8}$$

It turns out that the o.g.f. of A273596 has a simple expression as a white diamond product of power series, namely

$$\frac{1}{1-x} \lozenge \frac{1}{1-x} \quad = \quad 1+3x+9x^2+32x^3+139x^4+\cdots \tag{9}$$

(but note the offset is zero here instead of the offset of 2 used in the OEIS entry).

Proof of (9) . In the expansion

$$\frac{1}{1-x} \lozenge \frac{1}{1-x} \quad = \quad \sum_{i,j\geq 0} x^i \lozenge x^j, \tag{10}$$

we make use of (8) to extract the coefficient of x^n in the sum. After a short calculation we arrive at the result

$$\begin{aligned}[x^n]\left(\frac{1}{1-x} \lozenge \frac{1}{1-x}\right) \quad &= \quad \sum_{k=0}^{n}\sum_{i=0}^{n} \frac{(n-k)!i!}{n!}\binom{n}{i}\binom{i}{k} \\ &= \quad \sum_{k=0}^{n} f(n,k) \end{aligned} \tag{11}$$

where $f(n,k)$ denotes the inner sum:

$$f(n,k) \quad = \quad \sum_{i=0}^{n} \frac{(n-k)!i!}{n!}\binom{n}{i}\binom{i}{k}. \tag{12}$$

The Maple code

```
with(sumtools):
sumrecursion((n-k)!i!binomial(n, i)binomial(i, k)/n!, i, f(n));
```

returns the recurrence equation

$$f(n,k) \quad = \quad (n+1)f(n-1,k) - (n-1-k)f(n-2,k). \qquad (13)$$

Here $n \geq k$ and from (12) the initial conditions of the recurrence are $f(k,k) = 1$ and $f(k+1,k) = k+2$.

Now by (2)

$$\begin{aligned} a(n) &= \sum_{t=2}^{n}\sum_{r=1}^{t-1}(n-t)!\binom{n-r-1}{t-r-1}\binom{n-t+r-1}{r-1} \\ &= \sum_{t=2}^{n}\sum_{r=1}^{n-1}(n-t)!\binom{n-r-1}{t-r-1}\binom{n-t+r-1}{r-1} \\ &= \sum_{r=1}^{n-1}\sum_{t=2}^{n}(n-t)!\binom{n-r-1}{t-r-1}\binom{n-t+r-1}{r-1} \\ &= \sum_{r=1}^{n-1} g(n,r) \qquad (14) \end{aligned}$$

where $g(n,r)$ denotes the inner sum:

$$g(n,r) \quad = \quad \sum_{t=2}^{n}(n-t)!\binom{n-r-1}{t-r-1}\binom{n-t+r-1}{r-1}. \qquad (15)$$

The Maple code

```
with(sumtools):
sumrecursion((n-t)!binomial(n-r-1,t-r-1)binomial(n-t+r-1,r-1),t,g(n));
```

returns the recurrence equation

$$g(n,r) \quad = \quad (n-1)g(n-1,r) - (n-2-r)g(n-2,r). \qquad (16)$$

Here $n > r$ and the initial conditions of the recurrence calculated from (15) are

$$g(r+1,r) = 1 \text{ and } g(r+2,r) = r+1. \qquad (17)$$

If we now define $G(n,k) = g(n+2,k+1)$ it follows from (16) that $G(n,k)$ satisfies the recurrence

$$G(n,k) \quad = \quad (n+1)G(n-1,k) - (n-k-1)G(n-2,k) \qquad (18)$$

for $n \geq k$, with the initial conditions by (17) of $G(k,k) = g(k+2,k+1) = 1$ and $G(k+1,k) = g(k+3,k+1) = k+2$.

Comparison with the recurrence (13) yields

$$G(n,k) \quad = \quad f(n,k),$$

that is,

$$g(n+2,k+1) \quad = \quad f(n,k).$$

Hence from (14)

$$\begin{aligned} a(n+2) &= \sum_{k=1}^{n+1} g(n+2,k) \\ &= \sum_{k=0}^{n} g(n+2,k+1) \\ &= \sum_{k=0}^{n} f(n,k) \\ &= [x^n]\left(\frac{1}{1-x} \lozenge \frac{1}{1-x}\right) \end{aligned}$$

by (11) and the proof is completed. □

The asymptotic growth of the sequence is given by $a(n) \sim \exp(2) * n!/n^2$. This is too rapid for the generating function of the sequence to be a rational function. Thus this example shows that the white diamond product of a pair of rational power series may not be a rational power series.

Finally, we note the white diamond product

$$\begin{aligned} \frac{1}{1-tx} \lozenge \frac{1}{1-x} &= 1 + (1+2t)\,x + \left(1+3t+5t^2\right)x^2 + \left(1+4t+11t^2+16t^3\right)x^3 \\ &\quad + (1+5t+19t^2+49t^3+65t^4)x^4 + \cdots \end{aligned} \tag{19}$$

is the bivariate generating function for the square array A143409 read by antidiagonals.

REFERENCES

[1] P. Bala, The white diamond product of power series uploaded to A048993