Maple-assisted proof of formula for A269604

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Let a "state" consist of a pair (x, v) where $x \in [0,..., 6]$ and $v \in [0,..., 7]$. We interpret this as saying that the current value is x and the last repeated value was v (or, if $v = 7$, there was no previous repeated value). We start in any initial state $(x, 7)$. The allowed transitions from (x, v) are to (y, v) if $y \neq x$, or to (x, x) if $|x - v| > 1$ or $v = 7$. Listing the states as $s_i = (x, v)$ where $x = (i - 1) \bmod 7$ and $i = 1 + x + 7v$, $i = 1 \ldots 56$, we have the 56×56 transition matrix constructed below:

```
> with(LinearAlgebra):
  T:= Matrix(56,56):
  for x from 0 to 6 do
    for v from 0 to 7 do
      i:= 1 + x + 7*v;
      for y in {$0..6} minus {x} do
        T[i,1+y+7*v]:= 1;
      od:
      if abs(x-v) > 1 or v=7 then T[i,1+x+7*x]:= 1 fi
  od od:
```

Then $a(n) = \sum_{i=50}^{56} \sum_{j=1}^{56} (T^{n-1})_{ij} = u^T T^{n-1} v$ where u is the vector with the last 7 entries 1 and the others 0, while v is the vector of all 1's. To check, here are the first few values.

```
> u:= Vector([0$49,1$7]): v:= Vector(56,1):
> Tv[1]:= v: for n from 2 to 30 do Tv[n]:= T . Tv[n-1] od:
  seq(u^%T . Tv[n], n=1..30);
```

7, 49, 336, 2298, 15630, 105892, 714874, 4811578, 32300252, 216337084, 1446056046, 9648789758, 64281141440, 427655897226, 2841661493142, 18861464959350, 125070420653458, 828618463551536, 5485481885293294, 36288577806336542, 239911428612782620, 1585203987990332820, 10468802633545245370, 69104811390683250386, 455972292424876085692, 3007495303021614167374, 19830033526669337983286, 130709749833856866009110, 861333241339265788820254, 5674459633292167991281060 **(1)**

Here is the empirical recurrence formula. It says that $u^T Q(T)\, T^{n-1} v = 0$ for all nonnegative integers n, where Q is the following polynomial.

```
> n:= 'n': empirical:= a(n) = 26*a(n-1) - 243*a(n-2) + 833*a(n-3) +
  567*a(n-4) - 7567*a(n-5) - 1006*a(n-6) + 27361*a(n-7) + 31306*a
  (n-8) + 9984*a(n-9):
  Q:= unapply(add(coeff((lhs-rhs)(empirical),a(n-i))*t^(9-i),i=0.
  .9),t);
```

$$Q := t \mapsto t^9 - 26 t^8 + 243 t^7 - 833 t^6 - 567 t^5 + 7567 t^4 + 1006 t^3 - 27361 t^2 - 31306 t - 9984 \quad \textbf{(2)}$$

In fact, it turns out that $u^T Q(T) = 0$, so this is true.

```
> uT[0]:= u^%T:
  for n from 1 to 9 do uT[n]:= uT[n-1] . T od:
```

```
uQ:= add(coeff(Q(t),t,n)*uT[n],n=0..9):
> uQ;
```

[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] **(3)**

This completes the proof.