

Immersed curves and embedded bouquets

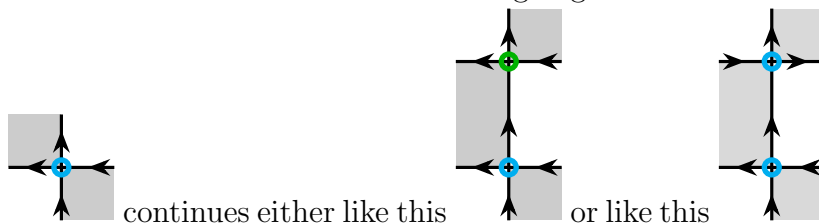
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$A_{268562}(n)$, the number of bicolored immersions of an oriented circle into an oriented surface with n double points (enumerated by Coquereaux and Zuber), is also the number of bicolored oriented directed embeddings of n -arc directed bouquets, denoted $\alpha^I(n, 2)$ in Section 5 of the paper by Ellingham and Ellis-Monaghan.

Bouquets are the single-vertex maps on surfaces; “directed embedding” means that each face of the map is bounded by a directed cycle, which is equivalent to the requirement that the directions of edges (incoming/outgoing) alternate as we go around the vertex. Bouquets are in bijection with chord diagrams, in which chords represent the edges of the bouquet (they are directed, since this is a directed bouquet), and the directed embedding condition is equivalent to the requirement that the directions of chords alternate as we go around the circle. In a diagram corresponding to a bicolored bouquet, each chord is colored into one of two colors, say blue and red.

Immersed curves are in bijection with chord diagrams (for $n > 0$), in which circular arcs represent the edges of the curve between intersections, while chords represent the intersections themselves; the chords are directed, in order to distinguish between two possible circular orders of the half-edges around each intersection. In a bicolored immersion, the map arising from the immersed curve is 2-colored (say, into white and gray), which creates two types of intersections (because the curve is oriented); they can be represented in the chord diagram by assigning each chord one of the two colors, say blue and green, depending on the type of the intersection (at a blue intersection, the quadrants bounded by two outgoing half-edges and by two incoming half-edges are colored gray and the other two are colored white; and vice versa for a green intersection). The preservation of the color of the face when going around it in the map requires



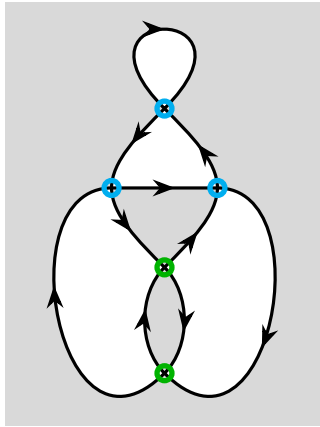
that a crossing like this continues either like this or like this, etc. This is equivalent to the following condition on the chord diagram arising from the bicolored immersion: when going around a circle, at every step either the direction of the chord (incoming/outgoing) or its color must change, but not both.

Since we consider oriented embedded bouquets and oriented immersed curves in oriented surfaces, and they are all unlabeled, the chord diagrams are considered equivalent under rotations but not turning over.

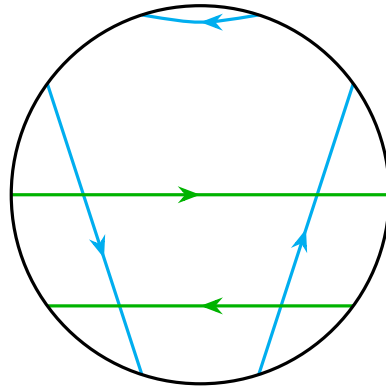
The bijection from the latter set of chord diagrams to the former set of chord diagrams is achieved by reversing the direction of each green chord and coloring it red.

For $n = 0$, there are two bicolored immersions of an oriented circle into the oriented sphere but only one chord diagram and only one bouquet.

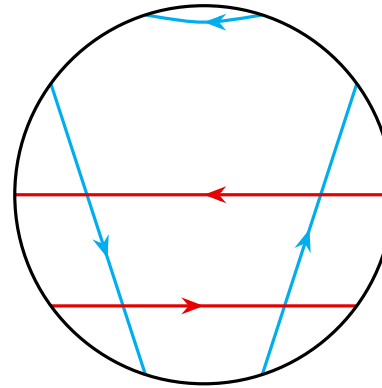
Example. A quadruple of objects related by the bijections described above, for $n = 5$.



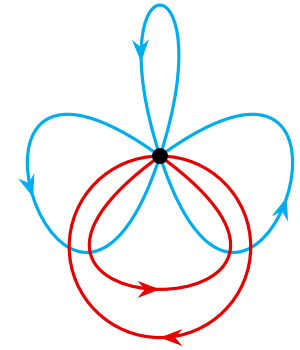
Bicolored immersion of a curve with 5 double points into a surface of genus ≥ 0



Blue-green chord diagram



Blue-red chord diagram



Bicolored directed embedding of a 5-arc bouquet into a surface of genus ≥ 1

$A_{268559}(n)$, the number of bicolored immersions of an unoriented circle into an oriented surface with n double points, is also the number of generic directed embeddings of n -arc directed bouquets, see (A6) in Section 5 of the paper by Ellingham and Ellis-Monaghan. Per that paper, this is also the number of bicolored orientable directed embeddings of n -arc directed bouquets. Indeed: both reversing the orientation of the surface in which a bouquet is embedded and reversing the direction of an immersed curve are equivalent to turning over the corresponding chord diagrams. In this case, the bijection works for $n = 0$ too.