Maple-assisted proof of empirical formula for A267243

Robert Israel

Sep 8 2019

Consider a "state" of the system to be a 1×11 binary array x, where $x_1 .. x_6$ represent a row, and for j from 7 to 11, $x_j = 1$ if this row and the previous ones have already determined that column $j - 5$ is lexicographically greater than column $j - 6$. In particular if $x_1 \ne x_2$ we must have $x_7 = 1$ and similarly for the others.
We enumerate the 486 possible states:

```
> S[2]:= [[0,0,0],[0,0,1],[0,1,1],[1,0,1],[1,1,0],[1,1,1]]:
  for i from 3 to 6 do
    S[i]:= map(proc(t) [op(t[1..i-1]),t[i-1],op(t[i..-1]),0], [op(t
  [1..i-1]),t[i-1],op(t[i..-1]),1],
       [op(t[1..i-1]),1-t[i-1],op(t[i..-1]),1] end proc, S[i-1])
  od:
  states:= S[6]:
  nops(states);
```

486 **(1)**

Although it is not part of the $n \times 6$ array, we may imagine that we start in state [0,0,0,0,0,0,0,0,0,0,0].
Let T be the 486×486 transition matrix where $T_{ij} = 1$ if state j can be followed by state i.

```
> T:= Matrix(486,486,proc(i,j) local k;
    if add(states[j,k]-states[i,k],k=1..6) > 0 then return 0 fi;
    for k from 7 to 11 do if states[j,k]>states[i,k] then return 0
  fi od;
    for k from 1 to 5 do if states[i,k]>=states[i,k+1] and states
  [j,k+6]<>states[i,k+6] then return 0 fi od;
  1
  end proc):
```

Then we should have $a_n = u^T T^n e$ where $u = (1, ..., 1)^T$ and $e = (1, 0,..., 0)^T$. To check, we compute the first few terms of the sequence. .

```
> E:= Vector(486): E[1]:= 1:
  U[0]:= Vector[row](486,1):
  for k from 1 to 25 do U[k]:= U[k-1].T od:
  seq(U[j] . E, j=1..25);
```

7, 50, 475, 6292, 107015, 2093467, 43555569, 924051709, 19614050515, 413556580944, 8645774602327, 179276181587698, 3691120876565687, 75550095426967737, 1538986699132717645, 31229753343696948035, 631791852881928155235, 12750422028897904475590, 256826148905935550268931, 5165267718138616456601312, 103758670192017224788929223, 2082304954694800745278445671, 41758046536776451033631049641, 836914729711338718829412150073, 16765720177011053621132904185811 **(2)**

Now the empirical formula is

```
> Emp:=  a(n) = 114*a(n-1) -5915*a(n-2) +186008*a(n-3) -3982785*a
```

```
(n-4) +61835542*a(n-5) -723657627*a(n-6) +6549515604*a(n-7)
-46652032035*a(n-8) +264676225246*a(n-9) -1205477853945*a(n-10)
+4427867737616*a(n-11) -13139368875011*a(n-12) +31468929403866*a
(n-13) -60602488003009*a(n-14) +93197329064964*a(n-15)
-113220771193368*a(n-16) +106920682204032*a(n-17)
-76630180181904*a(n-18) +40173465734208*a(n-19) -14497964755200*a
(n-20) +3213273369600*a(n-21) -329204736000*a(n-22):
```

This corresponds to $u^T P(T) T^n e = 0$ where $P(x)$ is the following polynomial of degree 22:

```
> P:= x^22 - add(coeff(rhs(Emp),a(n-i))*x^(22-i),i=1..22);
```

$$P := x^{22} - 114 x^{21} + 5915 x^{20} - 186008 x^{19} + 3982785 x^{18} - 61835542 x^{17} + 723657627 x^{16} - 6549515604 x^{15} + 46652032035 x^{14} - 264676225246 x^{13} + 1205477853945 x^{12} - 4427867737616 x^{11} + 13139368875011 x^{10} - 31468929403866 x^{9} + 60602488003009 x^{8} - 93197329064964 x^{7} + 113220771193368 x^{6} - 106920682204032 x^{5} + 76630180181904 x^{4} - 40173465734208 x^{3} + 14497964755200 x^{2} - 3213273369600 x + 329204736000 \quad \textbf{(3)}$$

It turns out that $u^T P(T) = 0$. The verification of this completes the proof.

```
> UP:= add(coeff(P,x,j)*U[j],j=0..22):
  UP . UP^%T;
```

$$0 \quad \textbf{(4)}$$