A 4-parameter family of embedded Riordan arrays

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Let $R = (f(x), xg(x))$ be a proper Riordan array. We shall associate with R a bi-infinite array R^* (itself not a Riordan array), which contains R as a subarray. The main result of this paper is to find a 4-parameter family of Riordan arrays that are embedded in R^*. For particular values of the parameters these arrays will in fact be embedded in the Riordan array R.

The generating function of the k^{th} column of R is, by definition, the Taylor series expansion of the series $f(x)(xg(x))^k$ about 0. In section 3 we prove a companion result for the rows of R: there is a pair of formal power series $F(x)$ and $G(x)$ such that the entries in n^{th} row of the proper Riordan array R, read from right to left, are the coefficients in the Taylor polynomial of degree n of the series $F(x)G(x)^n$ about 0. The proof uses the properties of a particular member of our family of Riordan arrays embeddded in R^*. In Section 4 we show that the Taylor series expansions about 0 of the functions $F(x)G(x)^n$, $n \in \mathbb{Z}$, are the generating functions for the rows of the extended array R^*.

1 Introduction

Let $f(x) = 1 + f_1x + f_2x^2 + \cdots$ and $g(x) = 1 + g_1x + g_2x^2 + \cdots$ be a pair of formal power series with (say) integer coefficients. The proper Riordan array

$$R = (R(n,k))_{n,k\geq 0} = (f(x), xg(x))$$

is defined as the lower unitriangular array whose k^{th} column has the ordinary generating function $f(x)(xg(x))^k$ [2, Section 2], [4, Section 1]. The elements of the array R are thus given by

$$R(n,k) = [x^n]f(x)(xg(x))^k \quad [n,k \geq 0]. \tag{1}$$

where $[x^n]$ denotes the coefficient extraction operator. The most well-known example of a proper Riordan array is Pascal's triangle of binomial coefficients $\left(\binom{n}{k}\right)_{n,k\geq 0}$, which is the Riordan array $\left(\frac{1}{1-x}, \frac{x}{1-x}\right)$.

We associate with the Riordan array R the bi-infinite array R^* (which is not a Riordan array) whose elements are given by

$$R^*(n,k) = [x^n]f(x)(xg(x))^k \quad [n,k \in \mathbb{Z}]. \tag{2}$$

R^* is an example of a recursive array [1]. We refer to the array R^* as the extended array associated with the Riordan array R.

The first few rows of R^* are shown below. Clearly, $R^*(n,k) = R(n,k)$ for $n, k \geq 0$, so we can view R as a subarray of R^* occupying the bottom right quadrant of R^*. We see by (2) that the entries in R^* lying above the main diagonal are all zero.

			$R^*(n,k)$,	$n,k \in \mathbb{Z}$						
		fg^{-3}	fg^{-2}	fg^{-1}		f	fg	fg^2	fg^3	$\cdots$
$n \diagdown k$	$\cdots$	-3	-2	-1	|	0	1	2	3	$\cdots$
$\vdots$	$\ddots$				|					
-3	$\cdots$	$R^*(-3,-3)$			|					
-2	$\cdots$	$R^*(-2,-3)$	$R^*(-2,-2)$		|					
-1	$\cdots$	$R^*(-1,-3)$	$R^*(-1,-2)$	$R^*(-1,-1)$	|					
-	-	-	-	-	-	-	-	-	-	-
0	$\cdots$	$R^*(0,-3)$	$R^*(0,-2)$	$R^*(0,-1)$	|	$R(0,0)$				
1	$\cdots$	$R^*(1,-3)$	$R^*(1,-2)$	$R^*(1,-1)$	|	$R(1,0)$	$R(1,1)$			
2	$\cdots$	$R^*(2,-3)$	$R^*(2,-2)$	$R^*(2,-1)$	|	$R(2,0)$	$R(2,1)$	$R(2,2)$		
$\vdots$		$\vdots$	$\vdots$	$\vdots$	|	$\vdots$	$\vdots$	$\vdots$	$\ddots$	

As an example we show part of the extended version of Pascal's triangle below. By (2), the array entries are given by

$$\begin{aligned} [x^n]\,\frac{1}{1-x}\cdot\frac{x^k}{(1-x)^k} &= \left[x^{n-k}\right](1-x)^{-k-1} \\ &= (-1)^{n-k}\binom{-k-1}{n-k} \quad [n,k \in \mathbb{Z}]. \end{aligned}$$

Note the bottom left quadrant of the array has all zero entries, which is usually not the case in an arbitrary extended array R^*.

$$\begin{pmatrix} \ddots & & & & & & & & & \\ \cdots & 1 & & & & & & & & \\ \cdots & -3 & 1 & & & & & & & \\ \cdots & 3 & -2 & 1 & & & & & & \\ \cdots & -1 & 1 & -1 & 1 & & & & & \\ & & & & & 1 & & & & \\ & & & & & 1 & 1 & & & \\ & & & & & 1 & 2 & 1 & & \\ & & & & & 1 & 3 & 3 & 1 & \\ & & & & & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

It follows from the definition of an extended array that the column generating functions of the extended version of Pascal's triangle are given by $1/(1-x)^n$ for $n \in \mathbb{Z}$. Note there is a similar result in this case for the rows of the extended array: if we read the rows from right to left, the row generating functions have the form $(1+x)^n$ for $n \in \mathbb{Z}$. In Section 4 we generalise this observation to a result on the row generating functions of the extended array R^* associated with an arbitrary proper Riordan array R.

2 Embedded Riordan arrays

In this section we are interested in finding Riordan arrays that are, so to speak, contained in either a given Riordan array R or in the associated extended array R^*. We refer to these arrays as embedded Riordan arrays of R or of R^*. One simple method to construct an embedded Riordan array of a proper Riordan array $R = (R(n,k)) = (f(x), xg(x))$ is by selecting columns from R. For example, the Riordan array $(f(x), xg^2(x)) = (R(n+k, 2k))_{n,k\geq 0}$ is constructed by taking the even indexed columns of R and arranging them in a lower unitriangular array. Similarly, the Riordan array $(f(x)g(x), xg^2(x))$ $= (R(n+k+1, 2k+1))_{n,k\geq 0}$ is constructed from the odd indexed columns of R. Embedded arrays of these two types have been studied in [3]. More generally, let $p \geq 0, q \geqq 1$ be integers. Then the Riordan array $(f(x)g^p(x), xg^q(x)) = (R(n+(q-1)k+p, qk+p))_{n,k\geq 0}$ is constructed from columns $p, p+q, p+2q, \ldots$ of R.

A less obvious example of an embedded Riordan array was given by Sprugnoli [9, Section 5.6]. The binomial coefficient array $\left(\binom{2n-k}{n}\right)$, which is A092392 in the database, is an embedded Riordan array of Pascal's triangle, constructed by taking the first $k+1$ elements from column k of Pascal's triangle and using them to form the k^{th} row of a lower unit triangular array:

$$\overset{\left(\binom{n}{k}\right)}{\begin{pmatrix} \underline{1} \\ 1 & \underline{1} \\ 1 & \underline{2} & \underline{1} \\ 1 & 3 & \underline{3} & \underline{1} \\ 1 & 4 & \underline{6} & \underline{4} & \underline{1} \\ 1 & 5 & 10 & \underline{10} & \underline{5} & \underline{1} \\ 1 & 6 & 15 & \underline{20} & \underline{15} & \underline{6} & \underline{1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}} \quad \rightarrow \quad \overset{\left(\binom{2n-k}{n}\right)}{\begin{pmatrix} 1 \\ 2 & 1 \\ 6 & 3 & 1 \\ 20 & 10 & 4 & 1 \\ 70 & 35 & 15 & 5 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}}$$

Using the Lagrange inversion formula, Sprugnoli shows that the array $\left(\binom{2n-k}{n}\right)$ is the Riordan array $(r'(x), r(x))$, where

$$r(x) \quad = \quad \text{Revert}(x(1-x)) = \frac{1-\sqrt{1-4x}}{2}.$$

By the above remarks we can then construct further Riordan arrays from the columns of A092392. For example, the array $\left(\binom{2n}{n+k}\right)$ formed from the even numbered columns of A092392 is the Riordan array $(r'(x), \frac{r^2(x)}{x})$. This is A094527. The array $\left(\binom{2n+1}{n+k+1}\right)$ formed from the odd numbered columns of A092392 is the Riordan array $(r'(x)\frac{r(x)}{x}, \frac{r^2(x)}{x})$. This is A111418 in the database.

Sprugnoli says his approach extends to deal with arrays of binomial coefficients of the form $\binom{pn+ak}{n-ck}$. Indeed, as we shall see in a moment, Sprugnoli's approach can be extended to find examples of embedded Riordan arrays in an arbitrary proper Riordan array. Given a Riordan array R and its associated extended array R^*, we shall construct a 4-parameter family of embedded Riordan arrays of R^*. For particular values of the parameters these arrays will be embedded in R, as was the case in the examples above. The proof uses series inversion. We shall make use of the following version of the Lagrange-Bürmann formula for formal power series [5, Theorem 1.2.4], [11]:

Let $f(x) = 1 + f_1x + f_2x^2 + f_3x^3 + \cdots$, $H(x) = h_0 + h_1x + h_2x^2 + h_3x^3 + \cdots$, be a pair of formal power series. Let $G(x) = \text{Revert}\left(\frac{x}{f(x)}\right)$. Then

$$[x^n]H(G(x)) \quad = \quad \frac{1}{n}\left[x^{n-1}\right]H^{'}(x)f(x)^n, \text{ for } n \geq 1. \tag{3}$$

The following result extends the calculations in [2, Theorem 6]. For related results see [10].

Theorem 1. *Let m, a, b, c be integers with $a > b$. Let $f(x) = 1 + f_1x + f_2x^2 + \cdots$ and $g(x) = 1 + g_1x + g_2x^2 + \cdots$ be a pair of formal power series. Let $R = (R(n,k))_{n,k\geq 0} = (f(x), xg(x))$ be the associated proper Riordan array and let R^* denote the extended array associated with R as defined in (2) above. Define the array $\widetilde{R} = \left(\widetilde{R}(n,k)\right)_{n,k\geq 0}$ by setting*

$$\widetilde{R}(n,k) \quad = \quad R^*((m+1)n - ak + c, mn - bk + c) \quad [n, k \geqq 0].$$

Then $\widetilde{R}$ is a Riordan array given by

$$\widetilde{R} \quad = \quad \left(f(r(x))g^c(r(x))\frac{xr'(x)}{r(x)}, x^{a-b}\frac{\left(\frac{r(x)}{x}\right)^{a-b}}{g^b(r(x))}\right), \tag{4}$$

where the power series $r(x)$ is determined by

$$r(x) \quad = \quad \mathit{Revert}\left(\frac{x}{g^m(x)}\right). \tag{5}$$

If m is nonzero then we can write $\widetilde{R}$ solely in terms of $f(x)$ and $r(x)$:

$$\widetilde{R} \quad = \quad \left(f(r(x))r'(x) \left(\frac{r(x)}{x} \right)^{\frac{c-m}{m}}, x^{a-b} \left(\frac{r(x)}{x} \right)^{a-b-\frac{b}{m}} \right) \quad [m \neq 0]. \tag{6}$$

Remark. If $a = b+1$, the array $\widetilde{R}$ is a proper Riordan array, while if $a > b+1$ the array $\widetilde{R}$ is a vertically stretched Riordan array.[1]

Proof. Let $n, k \geqq 0$. By (2)

$$\begin{aligned}
\widetilde{R}(n,k) &= R^*\left((m+1)n - ak + c, mn - bk + c\right) \\
&= \left[x^{(m+1)n-ak+c}\right] f(x)(xg(x))^{mn-bk+c} \\
&= [x^n]x^{(a-b)k} f(x)g(x)^{mn-bk+c} \\
&= (n+1)\left\{ \frac{1}{n+1}[x^n]\frac{f(x)}{g^{m-c}(x)} \left(\frac{x^{a-b}}{g^b(x)} \right)^k (g^m(x))^{n+1} \right\}.
\end{aligned} \tag{7}$$

Define a power series $H(x)$ by

$$H^{'}(x) \quad = \quad \frac{f(x)}{g^{m-c}(x)} \left(\frac{x^{a-b}}{g^b(x)} \right)^k, \text{ with } H(0) = 0. \tag{8}$$

Then (7) becomes

$$\begin{aligned}
\widetilde{R}(n,k) &= (n+1)\left\{ \frac{1}{n+1}[x^n]H^{'}(x)\,(g^m(x))^{n+1} \right\} \\
&= (n+1)[x^{n+1}]H(r(x)), \text{ by (3) and (5) since } n+1 \geq 1, \\
&= [x^n]\frac{d(H(r(x))}{dx} \\
&= [x^n]H^{'}(r(x))r'(x) \\
&= [x^n]\frac{f(r(x))}{g^{m-c}(r(x))} r'(x) \left(\frac{r(x)^{a-b}}{g^b(r(x))} \right)^k \quad \text{by (8)},
\end{aligned}$$

for $n, k \geq 0$.

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[1] A vertically stretched Riordan array $S = (f(x), x^s g(x))$, where s is a positive integer greater than 1, is defined as the lower triangular array whose k-th column has the ordinary generating function $f(x)(x^s g(x))^k$ - see [4, Section 2].

Thus $\widetilde{R}$ is the Riordan array

$$\widetilde{R} = (F(x), x^{a-b}G(x)),$$

where

$$F(x) = \frac{f(r(x))}{g^{m-c}(r(x))}r'(x), \quad G(x) = \frac{\left(\frac{r(x)}{x}\right)^{a-b}}{g^b(r(x))}. \tag{9}$$

Now it follows from (5) that

$$\frac{r(x)}{g^m(r(x))} = x. \tag{10}$$

Using (10) we can rewrite $F(x)$ as

$$F(x) = f(r(x))\, g^c(r(x))\, \frac{xr'(x)}{r(x)}.$$

Therefore $\widetilde{R}$ is the Riordan array

$$\widetilde{R} = \left(f(r(x))g^c(r(x))\frac{xr'(x)}{r(x)}, x^{a-b}\frac{\left(\frac{r(x)}{x}\right)^{a-b}}{g^b(r(x))} \right) \tag{11}$$

completing the proof of (4).

When $m \neq 0$ we can use (10) to rewrite (11) solely in terms of $f(x)$ and $r(x)$. We obtain

$$\widetilde{R} = \left(f(r(x))r'(x)\left(\frac{r(x)}{x}\right)^{\frac{c-m}{m}}, x^{a-b}\left(\frac{r(x)}{x}\right)^{a-b-\frac{b}{m}} \right) \quad [m \neq 0]$$

proving (6). $\square$

Example 1. In Theorem 1 take R to be Pascal's triangle $\left(\binom{n}{k}\right) = \left(\frac{1}{1-x}, \frac{x}{1-x}\right)$ so $f(x) = g(x) = \frac{1}{1-x}$ and choose $a = m, b = m-1$ and $c = 0$. Taking $m = 1, 2, ...$ we get a family of proper Riordan arrays $\left(\binom{2n-k}{n}\right)$, $\left(\binom{3n-2k}{2n-k}\right)$, $\left(\binom{4n-3k}{3n-2k}\right), ..., \left(\binom{(m+1)n-mk}{mn-(m-1)k}\right)$ embedded in Pascal's triangle and given by

$$\left(\frac{1}{1-r(x)}\frac{xr'(x)}{r(x)}, x\left(\frac{r(x)}{x}\right)^{\frac{1}{m}} \right),$$

where

$$r(x) = \text{Revert}\left(x(1-x)^m\right).$$

This result can be expressed in terms of Lambert's generalized binomial series $\mathcal{B}_t(x)$ [6, Sections 5.4 and 7.5] defined as

$$\mathcal{B}_t(x) \quad = \quad \sum_{n\geq 0} \frac{1}{nt+1}\binom{nt+1}{n}x^n.$$

Lambert's series satisfy the identity [6, equation 5.61], [7, Section 2]:

$$\sum_{n\geq 0}\binom{mn+k}{n}x^n \quad = \quad \frac{\mathcal{B}_m(x)^{k+1}}{m+(1-m)\mathcal{B}_m(x)}.$$

Hence the array of binomial coefficients $\left(\binom{(m+1)n-mk}{mn-(m-1)k}\right)_{n,k\geq 0}$ equals the proper Riordan array $\left(\frac{\mathcal{B}_{m+1}(x)}{m+1-m\mathcal{B}_{m+1}(x)}, x\mathcal{B}_{m+1}(x)\right)$.

3 A result on the row generating functions of Riordan arrays

Let $R = (R(n,k))_{n,k\geq 0} = (F(x), xG(x))$ be a proper Riordan array. By definition, the column generating functions of R are the Taylor series expansions about 0 of the functions $x^kF(x)G^k(x)$. We shall use a particular case of Theorem 1 to find a similar result expressing the generating functions of the rows of R as Taylor series expansions.

Applying Theorem 1 with $m = 1, a = 1$ and $b = c = 0$ tells us the array $\widetilde{R} = (R(2n-k,n))_{n,k\geq 0}$ is the proper Riordan array

$$\widetilde{R} \quad = \quad \left(xF(r(x))\frac{r'(x)}{r(x)}, x\left(\frac{r(x)}{x}\right)\right), \tag{12}$$

where

$$r(x) \quad = \quad \text{Revert}\left(\frac{x}{G(x)}\right). \tag{13}$$

The first few rows of the arrays R and $\widetilde{R}$ are shown below.

$$\overset{(R(n,k))}{\begin{pmatrix} \underline{R(0,0)} & & & & \\ R(1,0) & \underline{R(1,1)} & & & \\ R(2,0) & \underline{R(2,1)} & \underline{R(2,2)} & & \\ R(3,0) & R(3,1) & \underline{R(3,2)} & \underline{R(3,3)} & \\ \vdots & \vdots & \underline{R(4,2)} & \underline{R(4,3)} & \ddots \\ & & \vdots & \underline{R(5,3)} & \ddots \\ & & & \underline{R(6,3)} & \ddots \\ & & & \vdots & \ddots \end{pmatrix}} \rightarrow \overset{\widetilde{R} = (R(2n-k,n))}{\begin{pmatrix} R(0,0) & & & & \\ R(2,1) & R(1,1) & & & \\ R(4,2) & R(3,2) & R(2,2) & & \\ R(6,3) & R(5,3) & R(4,3) & R(3,3) & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}}$$

We observe that the entries in the n^{th} row of the Riordan array $\widetilde{R}$, when read from right to left, are the first $n+1$ entries from column n of the Riordan array $R = (F(x), xG(x))$, which has the generating function $F(x)G(x)^n$. Thus the entries in the n^{th} row of $\widetilde{R}$, when read from right to left, are simply the coefficients of the n^{th} degree Taylor polynomial of the function $F(x)G^n(x)$ about 0. In other words

$$\widetilde{R}(n,k) \quad = \quad \left[x^{n-k}\right] F(x)G(x)^n \quad [n \geq k \geq 0].$$

Now we claim that an arbitrary proper Riordan array $(f(x), xg(x))$ is equal to an array of the form $\widetilde{R} = (R(2n-k, n))_{n,k\geq 0}$ for some proper Riordan array $R = (F(x), xG(x))$, and hence the entries in the n^{th} row of the array $(f(x), xg(x))$ will be the coefficients of the n^{th} degree Taylor polynomial of the function $F(x)G^n(x)$ about 0. To prove the claim we see from (12) and (13) that we need to show that given power series $f(x) = 1 + f_1x + f_2x^2 + \cdots$ and $g(x) = 1 + g_1x + g_2x^2 + \cdots$ we can find power series $F(x)$ and $G(x)$ solving the following pair of equations:

$$xF(r(x))\frac{r'(x)}{r(x)} \quad = \quad f(x) \tag{14}$$

$$\frac{r(x)}{x} \quad = \quad g(x) \tag{15}$$

where

$$r(x) \quad = \quad \text{Revert}\left(\frac{x}{G(x)}\right). \tag{16}$$

From (15) and (16) we find

$$G(x) \quad = \quad \frac{x}{\text{Revert}(xg(x))}. \tag{17}$$

By (15)

$$r(x) \quad = \quad xg(x). \tag{18}$$

In (14), replace x with $\text{Revert}(r(x))$ and then use the identity

$$\frac{d\phi}{dx}\left(\phi^{-1}(x)\right) \quad = \quad \frac{1}{\frac{d\phi^{-1}}{dx}(x)}$$

for the derivative of an inverse function to obtain

$$\begin{aligned} F(x) \quad &= \quad \frac{x\frac{d}{dx}\left(\text{Revert}(r(x))\right)}{\text{Revert}(r(x))} f\left(\text{Revert}(r(x))\right) \\ &= \quad \frac{x\frac{d}{dx}\left(\text{Revert}(xg(x))\right)}{\text{Revert}(xg(x))} f\left(\text{Revert}(xg(x))\right). \end{aligned} \tag{19}$$

by (18).

It follows that the power series $F(x)$ and $G(x)$ given by (19) and (17) are such that the entries in row n of the Riordan array $(f(x), x, g(x))$ are the coefficients of the n^{th} degree Taylor polynomial of $F(x)G(x)^n$ about 0. For the sake of convenience we state this result in the form of a theorem.

Theorem 2. *Let $f(x) = 1 + f_1x + f_2x^2 + \cdots$ and $g(x) = 1 + g_1x + g_2x^2 + \cdots$ be a pair of formal power series. Let R be the proper Riordan array*

$$R = (R(n, k))_{n,k\geq 0} = (f(x), xg(x)),$$

where

$$R(n, k) = [x^n]f(x)(xg(x))^k \quad [n, k \geq 0].$$

Then there exists formal power series $F(x) = 1 + F_1x + F_2x^2 + \cdots$ and $G(x) = 1 + G_1x + G_2x^2 + \cdots$ defined by

$$F(x) = \frac{x\frac{d}{dx}\left(Revert(xg(x))\right)}{Revert(xg(x))} f\left(Revert(xg(x))\right)$$

$$G(x) = \frac{x}{Revert(xg(x))}$$

such that

$$R(n, k) = [x^{n-k}]F(x)G(x)^n \quad [n, k \geq 0]. \square$$

Example 2. The triangle A033184 in the OEIS is the Riordan array $(C(x), xC(x))$, where $C(x) = \frac{1-\sqrt{1-4x}}{2x} = 1 + x + 2x^2 + 5x^3 + 14x^4 + \cdots$ is the o.g.f. for the sequence of Catalan numbers A000108. The array begins

$$\begin{pmatrix} 1 & & & & & & \\ 1 & 1 & & & & & \\ 2 & 2 & 1 & & & & \\ 5 & 5 & 3 & 1 & & & \\ 14 & 14 & 9 & 4 & 1 & & \\ 42 & 42 & 28 & 14 & 5 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

It is easy to check the generating function $C(x)$ has the properties

$$\text{Revert}(xC(x)) = x - x^2$$

$$C\left(x-x^2\right) \quad = \quad \frac{1}{1-x}\,.$$

Then from (17) and (19) we find

$$F(x) \quad = \quad \frac{1-2x}{(1-x)^2}$$

and

$$G(x) \quad = \quad \frac{1}{1-x}\,.$$

Therefore by Theorem 2, the entries in row n of A033184 are the coefficients of the n^{th} degree Taylor polynomial of the rational function $F(x)G(x)^n = \frac{(1-2x)}{(1-x)^{n+2}}$ about 0. For example, for $n=5$ we have the Taylor expansion

$$\frac{(1-2x)}{(1-x)^7} \quad = \quad 1+5x+14x^2+28x^3+42x^4+42x^5+\mathrm{O}\left(x^6\right),$$

which gives row 5 of A033184 as $(42, 42, 28, 14, 5, 1)$.

The first few rows of the Riordan array $(F(x), xG(x)) = \left(\frac{1-2x}{(1-x)^2}, \frac{x}{1-x}\right)$ are shown below.

$$\begin{pmatrix}
\underline{1} & & & & & & & \\
0 & \underline{1} & & & & & & \\
-1 & \underline{1} & \underline{1} & & & & & \\
-2 & 0 & \underline{2} & \underline{1} & & & & \\
-3 & -2 & \underline{2} & \underline{3} & 1 & & & \\
-4 & -5 & -5 & \underline{5} & 4 & 1 & & \\
-5 & -9 & -14 & \underline{5} & 9 & 5 & 1 & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}$$

Exercise 1. With the conditions and notation as in Theorem 2, define a map ψ on pairs of powers series by $\psi : (f, g) \to (F, G)$. Show

$$\psi : \left(F, \frac{1}{G}\right) \quad \to \quad \left(f, \frac{1}{g}\right).$$

Exercise 2. A hitting-time array $H = (H(n,k))_{n,k\ge 0}$ is a proper Riordan array of the form $(xh'(x)/h(x), h(x))$, where $h(x) = x + h_2x^2 + h_3x^3 + \cdots$. Show the $(n,k)^{th}$ entry of the hitting-time array $(xh'(x)/h(x), h(x))$ is given by

$$H(n,k) \quad = \quad \left[x^{n-k}\right] G(x)^n \quad [n, k \ge 0],$$

where

$$G(x) = \frac{x}{\mathrm{Revert}(h(x))}\,.$$

This particular case of Theorem 2 is due to Peart and Woan [8, Theorem 4.1 (i)].

4 The row generating functions of the extended array R^*

Let $R = (f(x), xg(x))$ be a proper Riordan array and let R^* be the associated extended array. Theorem 2 tells us that there are power series F and G such that coefficients in the n^{th} degree Taylor polynomial of FG^n about 0 give the entries in the n^{th} row of R, for $n = 0, 1, 2, ...$ In this section we show that Taylor series of FG^n about 0 for $n \in \mathbb{Z}$ is, in fact, a generating function for the n^{th} row of the extended array R^*.

Theorem 3. *Let $f(x) = 1 + f_1x + f_2x^2 + \cdots$ and $g(x) = 1 + g_1x + g_2x^2 + \cdots$ be a pair of formal power series. Let R be the proper Riordan array*

$$R = (R(n,k))_{n,k\geq 0} = (f(x), xg(x)),$$

where

$$R(n,k) = [x^n]f(x)(xg(x))^k \quad [n,k \geq 0].$$

Let $R^* = (R^*(n,k))_{n,k\in\mathbb{Z}}$ be the extended array associated with R with entries defined by

$$R^*(n,k) = [x^n]f(x)(xg(x))^k \quad [n,k \in \mathbb{Z}].$$

Then there exists formal power series $F(x) = 1 + F_1x + F_2x^2 + \cdots$ and $G(x) = 1 + G_1x + G_2x^2 + \cdots$ defined by

$$F(x) = \frac{x\frac{d}{dx}\left(Revert(xg(x))\right)}{Revert(xg(x))} f\left(Revert(xg(x))\right)$$

$$G(x) = \frac{x}{Revert(xg(x))}$$

such that

$$R^*(n,k) = [x^{n-k}]F(x)G(x)^n \quad [n,k \in \mathbb{Z}].$$

Proof. Let m be a nonnegative integer. Define a subarray $R^*(m)$ of R^* by

$$R^*(m) = (R^*(n,k)) \quad [n,k \geq -m].$$

Thus $R^*(m)$ is the subarray of R^* starting at row $-m$ and column $-m$. Clearly, the array $R^*(m)$, when regarded as a lower unitriangular array, is the proper Riordan array $(f(x)g^{-m}(x), xg(x))$. In particular $R^*(0)$ is the array R. When $R^*(m)$ is viewed as an array in its own right, the row indices n and column indices k both start at 0. When $R^*(m)$ is regarded as a subarray of R^*, the n^{th} row of $R^*(m)$, $n = 0, 1, 2, ...$, gives the first $n + 1$ elements of the $(n - m)^{th}$ row of R^*.

We now apply Theorem 2 to the proper Riordan array $R^*(m)$ $= (f(x)g^{-m}(x), xg(x))$ to produce a pair of power series $\widetilde{F}(x), \widetilde{G}(x)$ given by

$$\widetilde{F}(x) = \frac{x\frac{d}{dx}(\text{Revert}(xg(x)))}{\text{Revert}(xg(x))}\widetilde{f}(\text{Revert}(xg(x))) \tag{20}$$

$$\widetilde{G}(x) = \frac{x}{\text{Revert}(xg(x))}, \tag{21}$$

where $\widetilde{f}(x) = f(x)g(x)^{-m}$, such that the coefficients of the n^{th} degree Taylor polynomial of $\widetilde{F}(x)\widetilde{G}(x)^n$ about 0 gives the entries in row n of $R^*(m)$, that is produces the first $n+1$ elements of the $(n-m)^{th}$ row of the extended array R^*.

It follows from (20) and (21) and the definitions of the functions $F(x)$ and $G(x)$ that

$$\widetilde{F}(x) = F(x)G(x)^{-m}, \quad \widetilde{G}(x) = G(x). \tag{22}$$

Hence, for $n = 0, 1, 2, ...$, the n^{th} degree Taylor polynomial of $\widetilde{F}(x)\widetilde{G}(x)^n = F(x)G(x)^{n-m}$ about 0 gives the first $n+1$ elements of the $(n-m)^{th}$ row of R^*. Therefore, setting $n = m+p$, we see that for $p \geq -m$ the $(m+p)^{th}$ degree Taylor polynomial of $F(x)G(x)^p$ about 0 gives the first $m+p+1$ elements of the p^{th} row of R^*. Letting m tend to infinity, we find that the Taylor expansion of $F(x)G(x)^p$ about 0 gives the elements of the p^{th} row of R^*. □

Example 3. Consider the proper Riordan array $R = (f(x), xg(x))$, where $f(x) = 1$ and $g(x) = 1-x$. This is A109466. The first few rows of the array R are shown below.

Column g.f.	f	fg	fg^2	fg^3	fg^4	fg^5	fg^6	$\cdots$	
$n\diagdown k$	0	1	2	3	4	5	6	$\cdots$	
0	1								F
1	0	1							FG
2	0	-1	1						FG^2
3	0	0	-2	1					FG^3
4	0	0	1	-3	1				FG^4
5	0	0	0	-3	-4	1			FG^5
6	0	0	0	1	6	-5	1		FG^6
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$

Using (17) and (19) we find

$$F(x) = \frac{1+\sqrt{1-4x}}{2\sqrt{1-4x}}, \quad G(x) = \frac{2x}{1-\sqrt{1-4x}}. \tag{23}$$

Therefore, by Theorem 2, the n^{th}degree Taylor polynomial of the function FG^n, $n = 0, 1, 2, ...$ about 0 gives the entries in row n of the Riordan array $R = (1, x(1-x))$. For example, for row 4 we have

$$F(x)G(x)^4 \quad = \quad 1 - 3x + x^2 + O(x^5), \tag{24}$$

giving correctly the five entries in row 4 of R, and hence also the first five entries in row 4 of the extended array R^*.

Below we show a subarray of the extended array R^*, starting at column $k' = -4$ and row $n' = -4$. Clearly, regarded as a lower unitriangular array, this subarray is the proper Riordan array $\big(f(x)g^{-4}(x), xg(x)\big)$, which we denote by $R^*(4)$. The n^{th} degree Taylor polynomial of the function $\widetilde{F}(x)\widetilde{G}(x)^n$ about 0, where by (22)

$$\widetilde{F}(x) = F(x)G(x)^{-4}, \quad \widetilde{G}(x) = G(x), \tag{25}$$

gives a row generating function for the n^{th} row of $R^*(4)$.

For example, for row 8 of the Riordan array $R^*(4)$ (corresponding to the beginning of row 4 of the extended arrayR^*) the row generating function is the Taylor polynomial of degree 8 of the function $\widetilde{F}(x)\widetilde{G}(x)^8 = F(x)G(x)^4$ about 0, that is the expansion

$$F(x)G(x)^4 \quad = \quad 1 - 3x + x^2 + x^5 + 7x^6 + 36x^7 + 165x^8 + O(x^9).$$

now gives the correct values for the nine entries in row 8 of $R^*(4)$, which are also the first nine entries in row 4 of R^*.

Array $R^*(4)$, $n, k \geq 0$

Column g.f.	fg^{-4}	fg^{-3}	fg^{-2}	fg^{-1}		f	fg	fg^2	fg^3	fg^4	$\cdots$	
n\k	0	1	2	3	|	4	5	6	7	8	$\cdots$	
0	1				|							FG^{-4}
1	4	1			|							FG^{-3}
2	10	3	1		|							FG^{-2}
3	20	6	2	1	|							FG^{-1}
-	-	-	-	-	-	-	-	-	-	-	-	-
4	35	10	3	1	|	1						F
5	56	15	4	1	|	0	1					FG
6	84	21	5	1	|	0	-1	1				FG^2
7	120	28	6	1	|	0	0	-2	1			FG^3
8	165	36	7	1	|	0	0	1	-3	1		FG^4
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	|	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$

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