σ(n) Equals the Area of the Symmetric Representation of σ(n)

Hartmut F. W. Hoft

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All references to lemmas, theorems, and definitions of notations are in the paper referenced in the LINK section of A241561.

DEFINITION

Consider a square of sidelength n with lower left-hand vertex at (0,0) and let s be a symmetric Dyck path from (0,n) to (n,0). We call the polygon D(s) defined by its sides of length n on the x- and y-axes and the symmetric Dyck path s a *Dyck-truncated square of sidelength n* and its *area* A(s). When D(s_1) is wholly contained in D(s_2) we write D(s_1) $\subset$ D(s_2); s_1 and s_2 may share vertices or parts of sides.

When a symmetric Dyck path s has an odd (even) number of legs from its start to its center vertex, then its two central legs form a vertex pointing away from (towards) the origin so that a square can be fitted inside (outside) D(s) at that center vertex. When the sidelength of the square to be fitted at the center vertex is smaller than the length of the last leg of the symmetric Dyck path the resulting boundary is a symmetric Dyck path with the two additional legs now meeting on the diagonal.

THEOREM

For all n ≥ 1, σ(n) = A(a(n+1,row(n+1))) - A(a(n,row(n))), in other words, $\sigma$the area of the symmetric representation of σ(n) equals σ(n), the sum of divisors of n.

Consider rows n and n+1 in the triangle of this sequence. We start with the first pair of squares of sidelengths n and n+1 at the origin. Then we successively fit squares of sidelength a235791(n,k+1) and a235791(n+1,k+1) inside or outside depending on the type of central vertex of the current respective Dyck-truncated squares D(a(n,k)) and D(a(n+1,k)). For an odd-numbered step k a square is fitted outside at the central vertex of the last Dyck-truncated square created, and for an even-numbered step a square is fitted inside from the central vertex. In either case the length of the k-th legs of the two new Dyck paths are a235791(n,k) - a235791(n,k+1) and a235791(n+1,k) - a235791(n+1,k+1). Obviously, the respective lengths of the two paths are not changed. Since the entries in a row of A235791 are strictly decreasing, two new symmetric Dyck paths with two additional legs are created with each step with legs 1 through k given by the first k entries in rows n and n+1 of A237591. The coordinates of the sequence of central vertices of the Dyck-truncated squares resulting from this process are given by the numbers in rows n and n+1 in this sequence. By this construction, the two completed symmetric Dyck paths a(n,row(n)) and a(n+1,row(n+1)) are those described in rows n and n+1 of A237593. Theorem 1 in the link of A241561 shows that the two symmetric Dyck paths never cross so that the area between them is the symmetric representation of σ(n+1).

The identity A(a(n,row(n)))) = $\sum_{i=1}^{\text{row}(n)} (-1)^{i+1} \times \text{a236104}(n, i) = \sum_{k=1}^{n} \sigma(k)$ as stated in A236104, for all n ≥ 1, now shows that the symmetric representation of σ(n+1) as defined by the symmetric Dyck paths a(n,row(n)) and a(n+1,row(n+1)) of A237593 equals σ(n+1).