A note on A262957 and A263295

Peter Bala, Dec 24 2015

Proposition. Let a_n be a sequence of integers with $a_n > 1$ for $n \geq 2$. Then the infinite alternating continued fraction $a_1 - \cfrac{1}{a_2} + \cfrac{1}{a_3} - \cfrac{1}{a_4} + \cdots$ converges to a real number with the simple continued fraction expansion

$$a_1 - 1 + \frac{1}{1} + \frac{1}{a_2 - 1} + \frac{1}{a_3 - 1} + \frac{1}{1} + \frac{1}{a_4 - 1} + \frac{1}{a_5 - 1} + \frac{1}{1} + \cdots .$$

Proof. Starting from the identity

$$a_1 - \frac{1}{a_2 + x} = a_1 - 1 + \frac{1}{1} + \frac{1}{a_2 - 1 + x}$$

a simple induction argument shows that for $n \geq 1$ we have

$$a_1 - \frac{1}{a_2} + \frac{1}{a_3} - \frac{1}{a_4} + \cdots + \frac{1}{a_{2n-1}} - \frac{1}{a_{2n}}$$

$$= a_1 - 1 + \frac{1}{1} + \frac{1}{a_2 - 1} + \frac{1}{a_3 - 1} + \frac{1}{1} + \cdots + \frac{1}{1} + \frac{1}{a_{2n-2} - 1} + \frac{1}{a_{2n-1} - 1} + \frac{1}{1} + \frac{1}{a_{2n} - 1} .$$

Letting $n \to \infty$, and recalling that infinite simple continued fractions always converge, completes the proof. □

Example. Let $a_n = n$. Then the alternating continued fraction

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

converges and has the simple continued fraction expansion $[0; 1, 1, 2, 1, 3, 4, 1, 5, 6, 1, ...]$ as observed by Mohamed Sabba in A262957.