A supercongruence for A262732

Peter Bala, Sep 24 2021

The terms of A262732 are defined by

$$a(n) = \frac{1}{n!} \frac{(5n)!}{\left(\frac{5n}{2}\right)!} \frac{\left(\frac{3n}{2}\right)!}{(3n)!} \tag{1}$$

It can be shown that

$$a(n) = \sum_{k=0}^{n} \binom{5n}{k} \binom{4n-k-1}{n-k}, \tag{2}$$

for example, by using Zeilberger's algorithm to verify that (1) and (2) satisfy the same linear recurrence

$a(n) = 20(5n-1)(5n-3)(5n-7)(5n-9)/(n(3n-1)(3n-3)(3n-5))a(n-2)$

and have the same initial conditions.

Proposition 1. *The supercongruence $a(p) \equiv 8 \pmod{p^3}$ holds for prime $p \geq 5$.*

Proof. Let $p \geq 5$ be prime. We make use of the binomial sum representation (2) for $a(p)$. We rewrite the sum by separating out the first ($k = 0$) summand and last ($k = p$) summand and adding together the k-th and $(p-k)$-th summands for $1 \leq k \leq \frac{p-1}{2}$ to obtain

$$a(p) = \binom{4p-1}{p} + \binom{5p}{p} + \sum_{k=1}^{\frac{p-1}{2}} \left(s(k) + s(p-k)\right),$$

where

$$s(k) = \binom{5p}{k} \binom{4p-k-1}{p-k}.$$

By [Mes, equation 35]

$$\binom{4p-1}{p} = \frac{3}{4} \binom{4p}{p} \equiv 3 \pmod{p^3} \quad \text{for prime } p \geq 5$$

and

$$\binom{5p}{p} \equiv 5 \pmod{p^3} \quad \text{for prime } p \geq 5.$$

Hence

$$a(p) \equiv 8 + \sum_{k=1}^{\frac{p-1}{2}} \left(s(k) + s(p-k)\right) \pmod{p^3}, \quad \text{prime } p \geq 5. \tag{3}$$

To establish the proposition we will show that each summand $s(k) + s(p-k)$ in (3) is divisible by p^3. One easily checks that

$$(p-k)!\,(s(k) + s(p-k)) = \binom{5p}{k} \frac{(3p+k-1)!}{(3p-1)!} \left(\frac{(5p-k)!}{(4p+k)!} + \frac{(4p-k-1)!}{(3p+k-1)!} \right). \tag{4}$$

We claim that each factor on the right side of (4) is divisible by p. Clearly, the first two factors are divisible by p for k in the summation range $1 \leq k \leq \dfrac{p-1}{2}$. The third factor is also divisible by p since

$$\begin{aligned} \frac{(5p-k)!}{(4p+k)!} + \frac{(4p-k-1)!}{(3p+k-1)!} &= (4p+k+1)(4p+k+2)\cdots(5p-k) \; + \\ &\qquad (4p-k-1)(4p-k-2)\cdots(3p+k) \\ &\equiv (k+1)(k+2)\cdots(k+r) + (-1)^r (k+1)(k+2)\cdots(k+r) \pmod{p} \\ &\equiv 0 \pmod{p} \end{aligned}$$

where $r = p - 2k$ is a positive odd integer.

It follows from (4) that p^3 divides $(p-k)!\,(s(k) + s(p-k))$ and since $(p-k)!$ is clearly not divisible by p for k in the indicated range we see that p^3 divides $s(k) + s(p-k)$, thus completing the proof of the proposition. □

Conjecture. We conjecture that the supercongruences

$$a\left(np^k\right) \equiv a\left(np^{k-1}\right) \pmod{p^{3k}} \tag{5}$$

hold for prime $p \geq 5$ and all $n, k \in \mathbb{N}$.

We can generalise Proposition 1 by considering sequences $(a_m(n))_{n \geq 0}$ defined by

$$a_m(n) = \sum_{k=0}^{n} \binom{mn}{k} \binom{(m-1)n-k-1}{n-k}. \tag{6}$$

The present sequence A262732 is the case $m = 5$. Then just as in the Proposition one can show that $a_m(p) \equiv a_m(1) \pmod{p^3}$ for integer m and prime $p \geq 5$. Cases in the OEIS include A000984 ($m = 2$), A091527 ($m = 3$), A001448 ($m = 4$), A211419 ($m = 6$), A262733 ($m = 7$) and A211421 ($m = 8$). We expect that the supercongruences (5) also hold in these cases.

References

[Mes] R. Mestrovic, Wolstenholme's theorem: Its Generalizations and Extensions in the last hundred and fifty years (1862-2011), arXiv:1111.3057 [math.NT], 2011.