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RICHARD J. MATHAR

ABSTRACT. This is an example of the exercise of transforming binomial expressions like [1, A262717] to Γ-functions to characterize the generating functions as Generalized Hypergeometric Functions, eventually to obtain a D-finite recurrence.

1. Hypergeometric reduction

Starting from

$$a(n) = \frac{n-1}{2n-1}\binom{3n-2}{n} + \frac{n+1}{2n+1}\binom{3n}{n} - \binom{3n-1}{n} \tag{1}$$

a first step is to represent the binomial coefficients with $n! = \Gamma(n+1)$ [2, 5.4.1] as ratios of Γ-functions and to assign a single Γ-product ratio to $a(n)$:

$$\begin{aligned} a(n) = -[&-2\Gamma(3n-1)\Gamma(2n+1)\Gamma(2n)n^2 + \Gamma(3n-1)\Gamma(2n+1)\Gamma(2n)n \\ &+ \Gamma(3n-1)\Gamma(2n+1)\Gamma(2n) - 2\Gamma(3n+1)\Gamma(2n-1)\Gamma(2n)n^2 \\ &-\Gamma(3n+1)\Gamma(2n-1)\Gamma(2n)n+\Gamma(3n+1)\Gamma(2n-1)\Gamma(2n)+4\Gamma(3n)\Gamma(2n-1)\Gamma(2n+1)n^2-\Gamma(3n)\Gamma(2n-1)\Gamma(2n+1)] \\ &/[(2n-1)(2n+1)\Gamma(n+1)\Gamma(2n-1)\Gamma(2n+1)\Gamma(2n)]. \end{aligned} \tag{2}$$

The fundamental equation $x\Gamma(x) = \Gamma(x+1)$ [2, 5.5.1] condenses these factors:

$$a(n) = \frac{1}{2}\frac{(n+3)(n-1)}{2n+1}\frac{\Gamma(3n-1)}{\Gamma(n+1)\Gamma(2n)} = \frac{1}{2}\frac{(n+3)(n-1)}{(2n+1)(3n-1)}\binom{3n-1}{n}, \quad (n \ge 1). \tag{3}$$

The generating function is by definition

$$g(x) = \sum_{n\ge 0} a(n)x^n = \frac{1}{2}\sum_{n\ge 0}\frac{(n+3)(n-1)}{2n+1}\frac{\Gamma(3n-1)}{\Gamma(2n)}\frac{x^n}{n!} \tag{4}$$

and we separate the first 2 terms to avoid the poles in the denominator:

$$g(x) = 1 + \frac{1}{2}\sum_{n\ge 2}\frac{(n+3)(n-1)}{2n+1}\frac{\Gamma(3n-1)}{\Gamma(2n)}\frac{x^n}{n!}. \tag{5}$$

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Reduction to Pochhammer Symbols reads [3, 4]

(6)
$$g(x) = 1+\frac{1}{4}\sum_{n\ge 2}\frac{(n+3)(n-1)}{n+1/2}\frac{\Gamma(3n-1)}{\Gamma(2n)}\frac{x^n}{n!} = 1+\frac{1}{4}x^2\sum_{n\ge 0}\frac{(n+5)(n+1)}{n+5/2}\frac{\Gamma(3n+5)}{\Gamma(2n+4)}\frac{x^n}{(n+2)!}$$
$$= 1+\frac{1}{4}x^2\sum_{n\ge 0}\frac{(n+5)}{(n+2)(n+5/2)}\frac{\Gamma(3n+5)}{\Gamma(2n+4)}\frac{x^n}{n!} = 1+\frac{1}{4}x^2\sum_{n\ge 0}\frac{5(6)_n(2)_n(5/2)_n}{(5)_n2(3)_n(5/2)(7/2)_n}\frac{\Gamma(5)(5)_{3n}}{\Gamma(4)(4)_{2n}}\frac{x^n}{n!}$$
$$= 1+x^2\sum_{n\ge 0}\frac{(6)_n(2)_n(5/2)_n}{(5)_n(3)_n(7/2)_n}\frac{(5)_{3n}}{(4)_{2n}}\frac{x^n}{n!} = 1+x^2\sum_{n\ge 0}\frac{(6)_n(2)_n(5/2)_n}{(5)_n(3)_n(7/2)_n}\frac{(5/3)_n(6/3)_n(7/3)_n3^{3n}}{(4/2)_n(5/2)_n2^{2n}}\frac{x^n}{n!}$$
$$= 1 + x^2\sum_{n\ge 0}\frac{(6)_n}{(5)_n(3)_n(7/2)_n}(5/3)_n(6/3)_n(7/3)_n\frac{(27x/4)^n}{n!}$$
$$= 1 + x^2 {}_4F_3\left(\begin{matrix}6, 5/3, 2, 7/3\\ 5, 3, 7/2\end{matrix} \mid \frac{27x}{4}\right).$$

The penultimate line means

$$a_{n+2} = \frac{(6)_n(5/3)_n(2)_n(7/3)_n(27/4)^n}{(5)_n(3)_n(7/2)_n n!} \tag{7}$$

and demonstrates the 'standard' D-finite recurrence for the coefficients obtained for Generalized Hypergeometric Series, employing $(\alpha)_n/(\alpha)_{n-1} = \alpha + n - 1$:

(8)
$$\frac{a_{n+2}}{a_{n+1}} = \frac{(6)_n(5/3)_n(2)_n(7/3)_n(27/4)^n}{(5)_n(3)_n(7/2)_n n!}\times\frac{(5)_{n-1}(3)_{n-1}(7/2)_{n-1}(n-1)!}{(6)_{n-1}(5/3)_{n-1}(2)_{n-1}(7/3)_{n-1}(27/4)^{n-1}}$$
$$= \frac{(6)_n(5/3)_n(2)_n(7/3)_n 27/4}{(5)_n(3)_n(7/2)_n n}\times\frac{(5)_{n-1}(3)_{n-1}(7/2)_{n-1}}{(6)_{n-1}(5/3)_{n-1}(2)_{n-1}(7/3)_{n-1}}$$
$$= \frac{(5+n)(2/3+n)(1+n)(4/3+n)27/4}{(4+n)(2+n)(5/2+n)n} = \frac{3(5+n)(2+3n)(1+n)(4+3n)}{2(4+n)(2+n)(5+2n)n}.$$

With in index shift of 2 this is the formula of 2017:

$$\frac{a_n}{a_{n-1}} = \frac{3(n+3)(3n-4)(n-1)(3n-2)}{2(n+2)n(2n+1)(n-2)}. \tag{9}$$

References

1. O. E. I. S. Foundation Inc., *The On-Line Encyclopedia Of Integer Sequences*, (2023), https://oeis.org/. MR 3822822
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3. Ranjan Roy, *Binomial identities and hypergeometric series*, Amer. Math. Monthly **94** (1987), no. 1, 36–46. MR 0873603
4. Lucy Joan Slater, *Generalized hypergeometric functions*, Cambridge University Press, 1966. MR 0201688

URL: `http://www.mpia.de/~mathar`

Hoeschstr. 7, 52372 Kreuzau, Germany