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### FURTHER IDENTITIES AND CONGRUENCES FOR THE COEFFICIENTS OF MODULAR FORMS

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I. If  $n$  is a non-negative integer, define  $p_r(n)$  by

$$\sum p_r(n)x^n = \prod (1 - x^n)^r;$$

otherwise define  $p_r(n)$  as 0. (Here and in what follows all sums will be extended from 0 to  $\infty$  and all products from 1 to  $\infty$  unless otherwise stated.)  $p_r(n)$  is thus generated by the powers of  $x^{-1/24}\eta(\tau)$ , where

$$\eta(\tau) = \exp(\pi i\tau/12) \prod (1 - x^n), x = \exp 2\pi i\tau,$$

is the Dedekind modular form. In (1) it was shown that recurrence formulas for these coefficients depending on a parameter  $p$ ,  $p$  a prime, exist for all positive integral  $r$ . The number of terms in these recurrence formulas is in general a function of  $r$  and  $p$ , which is determined in (1). If  $r$  is even,  $0 < r \leq 26$ , it was shown in (2), (3) that three term recurrence formulas exist for these coefficients for  $p$  satisfying appropriate congruence conditions with respect to  $24$  as modulus. These include, for example, Mordell's identity for  $\tau(n) = p_{24}(n - 1)$ :

$$\tau(np) = \tau(n)\tau(p) - p^{11}\tau(n/p).$$

$p_r(n)$  bears some relation to the function  $q_r(n)$ , the number of representations of  $n$  as a sum of  $r$  squares. If

$$n = \sum_{k=1}^r \frac{1}{2}(3x_k^2 \pm x_k)$$

is a representation of  $n$  as a sum of  $r$  pentagons, then  $p_r(n)$  is the excess of the number of those representations of  $n$  in which

$$\sum_{k=1}^r x_k$$

is even over those in which it is odd. Since the associated modular form is of fractional dimension when  $r$  is odd and of integral dimension when  $r$  is even, identities for odd  $r$  lie deeper than identities for even  $r$ ; and indeed quadratic reciprocity symbols appear. A good example is furnished by the identity

$$(1) \quad q_3(np^2) = \left\{ p + 1 - \left( \frac{-n}{p} \right) \right\} q_3(n) - \left\{ p - \left( \frac{-n}{p} \right) \right\} q_3\left( \frac{n}{p^2} \right)$$

given by G. Pall in (7).

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In this paper we study the coefficients  $p_r(n)$  for  $r$  odd,  $0 < r < 24$ . We shall demonstrate the existence of identities of type (1) for all primes  $p > 3$ , and for  $p = 3$  when  $r$  is a multiple of 3. Most of the discussion that follows depends upon (1), and we assume familiarity with the contents of this paper.

After this paper was written the author received from J. H. van Lint a copy of his dissertation, "Hecke Operators and Euler Products" (October 1957, University of Utrecht), which contains a proof of formulas (5) and (11) of the next section. (There are minor inaccuracies in van Lint's expression for formula (5).) van Lint's proof is based upon properties of modular forms while the author's is based upon properties of modular functions. The methods are quite different and yield different results in general.

II. Let  $p$  be a prime. If  $g(\tau)$  is a function on  $\Gamma_0(p)$ , we say that  $g(\tau)$  is *entire* if it is regular in the interior of the upper  $\tau$  half-plane and has polar singularities at most in appropriate uniformizing variables at the two parabolic vertices  $\tau = 0, i\infty$  of the fundamental region of  $\Gamma_0(p)$ . We require the following lemma:

LEMMA 1. *If  $g(\tau)$  is a function on  $\Gamma_0(p)$ , then so is  $g(-1/p\tau)$ . If in addition  $g(\tau)$  is entire, then so is  $g(-1/p\tau)$ .*

*Proof.* The second statement is clear, since the substitution  $\tau' = -1/p\tau$  permutes the parabolic points  $\tau = 0, i\infty$  and takes interior points of the upper  $\tau$  half-plane into interior points of the upper  $\tau$  half-plane. To prove the first, let

$$M = \begin{bmatrix} a & b \\ pc & d \end{bmatrix}$$

belong to  $\Gamma_0(p)$ , and let

$$T_p = \begin{bmatrix} 0 & -1 \\ p & 0 \end{bmatrix}$$

be the matrix of the transformation  $\tau' = -1/p\tau$ . Then

$$T_p M T_p^{-1} = \begin{bmatrix} d & -c \\ -pb & a \end{bmatrix} = M_0,$$

where  $M_0$  also belongs to  $\Gamma_0(p)$ .

Suppose now that  $g(\tau)$  is a function on  $\Gamma_0(p)$ , and put  $f(\tau) = g(-1/p\tau) = g(T_p\tau)$ . Then  $f(M\tau) = g(T_p M\tau) = g(M_0 T_p\tau) = g(T_p\tau) = f(\tau)$ , so that  $f(\tau)$  is also a function on  $\Gamma_0(p)$ . The lemma is therefore proved.

As in (1) we write  $T_p g(\tau) = g(T_p\tau)$ .

Following the notation of (1), let  $p$  be a prime  $> 3$ , and  $Q$  a power of  $p$ . Define

$$\epsilon = \begin{cases} p & Q \text{ a square} \\ 1 & \text{otherwise,} \end{cases}$$

and set

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$$h(\tau) = \frac{\eta(pQ\tau)}{\eta(\epsilon\tau)}$$

Let

$$R_n = \begin{bmatrix} 1 & 0 \\ -np & 1 \end{bmatrix}$$

Then if  $r$  is an integer, it is shown in (1) that the function

$$F(r, p, Q; \tau) = \sum_{n=0}^{Q-1} h^r(R_n\tau)$$

is an entire modular function on  $\Gamma_0(p)$ . Define

$$G(r, p, Q; \tau) = T_p F(r, p, Q; \tau)$$

By Lemma 1,  $G(r, p, Q; \tau)$  is also an entire modular function on  $\Gamma_0(p)$ . It is shown in (1) that

$$G(r, p, Q; \tau) = \left(\frac{pQ}{\epsilon}\right)^{-r/2} \eta^{-r}\left(\frac{p\tau}{\epsilon}\right) \sum_{n=0}^{Q-1} \eta^r\left(\frac{\tau + 24n}{Q}\right)$$

We write  $n:Q$  in a summation to indicate that  $n$  runs over a reduced set of residues mod  $Q$ . We shall prove the following lemma:

LEMMA 2. Suppose that  $Q$  is a square, and put  $Q' = Q/p$ . Then

$$F(r, p, Q; \tau) + G(r, p, Q'; \tau) = F(r, p, Q'; p\tau) + G(r, p, Q; p\tau)$$

Proof. Put

$$g_n = h^r(R_n\tau) = \left\{ \frac{\eta(pQR_n\tau)}{\eta(pR_n\tau)} \right\}^r$$

Then

$$F(r, p, Q; \tau) = \sum_{n=0}^{Q-1} g_n = \sum_{n:Q} g_n + \sum_{n=0}^{Q'-1} g_{np}$$

Now

$$\frac{\eta(pQR_{np}\tau)}{\eta(pR_{np}\tau)} = \frac{\eta(pQ'R_n p\tau)}{\eta(R_n p\tau)},$$

which implies that

$$F(r, p, Q; \tau) = \sum_{n:Q} g_n + F(r, p, Q'; p\tau)$$

Thus we need only consider  $\sum_{n:Q} g_n$ . This sum is treated in (1), where it is shown by means of the transformation formula for the Dedekind  $\eta$ -function that

$$\sum_{n:Q} g_n = Q^{-r/2} \eta^{-r}(p\tau) \sum_{n:Q} \eta^r\left(\frac{p\tau + 24n}{Q}\right)$$

Transforming this sum by means of the identity

$$\sum_{n:Q} f(n) = \sum_{n=0}^{Q-1} f(n) - \sum_{n=0}^{Q'-1} f(np)$$

we find easily that

$$\sum_{n:Q} g_n = G(r, p, Q; p\tau) - G(r, p, Q'; \tau).$$

The lemma is thus proved.

The functions so defined are also entire modular functions on  $\Gamma_0(p)$  when  $p = 3$ , if  $r$  is a multiple of 3. We assume from now on that  $r$  is odd,  $0 < r < 24$ ; and that  $p$  is a prime such that  $p > 3$  when  $(r, 3) = 1$  and  $p > 2$  when  $3|r$ . We put

$$\nu = \frac{(p^2 - 1)}{24}, \mu = \left[ \frac{r\nu}{p} \right], \delta = r\nu - p\mu;$$

and define

$$\alpha_p = \begin{cases} 1 & p \equiv 1 \pmod{4} \\ i & p \equiv 3 \pmod{4} \end{cases}.$$

LEMMA 3. The function

$$f = F(r, p, p^2; \tau) + G(r, p, p; \tau)$$

is constant.

*Proof.* From (3), formula (2.5.2) and (1), page 354 we have

$$(2) \quad F(r, p, p; \tau) = x^{r\nu} \prod (1 - x^{np^2})^r (1 - x^n)^{-r} + a p^{(1-r)/2} \prod (1 - x^n)^{-r} \sum \binom{r\nu - n}{p} p_r(n) x^n,$$

where  $a = \alpha_p \exp \{-i\pi r(p-1)/4\}$ , and

$$\binom{r\nu - n}{p}$$

is the Legendre-Jacobi symbol of quadratic reciprocity; and

$$(3) \quad G(r, p, p; \tau) = p^{1-r} x^{-\mu} \prod (1 - x^{np})^{-r} \sum p_r(np + \delta) x^n.$$

Similarly, from (1, p. 354) we have (since  $r\nu < p^2$ )

$$(4) \quad G(r, p, p^2; \tau) = p^{2-r} \prod (1 - x^n)^{-r} \sum p_r(np^2 + r\nu) x^n.$$

(We take this opportunity to correct an error in the second displayed formula for  $T_p F$  on page 354 of (1). The coefficient should be  $Q(pQ/\epsilon)^{-r/2}$  instead of  $p(pQ/\epsilon)^{-r/2}$ .)

From Lemma 2 with  $Q = p^2$  we have that

$$f = F(r, p, p; p\tau) + G(r, p, p^2; p\tau),$$

which is regular at  $\tau = i\infty$  by formulas (2) and (4). In addition,

$$T_n f = F(r, p, p; \tau) + G(r, p, p^2; \tau)$$

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so that (2) and (4) imply that  $f$  is regular at  $\tau = 0$  as well. Since  $f$  is an entire modular function on  $\Gamma_0(p)$ , this implies that  $f$  is constant, proving the lemma.

If we consider the expansion of  $T_n f$  in powers of  $x$  as in (1) we obtain our principal result, by comparing coefficients of like powers of  $x$ :

THEOREM 1. For all integral  $n$ ,

$$(5) \quad p_r(np^2 + rv) - \gamma_n p_r(n) + p^{\tau-2} p_r\left(\frac{n - rv}{p^2}\right) = 0,$$

where

$$\gamma_n = c - \left(\frac{rv - n}{p}\right) p^{(r-3)/2} a \quad \text{and} \quad c = p_r(rv) + \left(\frac{rv}{p}\right) p^{(r-3)/2} a.$$

If in this identity  $n$  is replaced by  $np + \delta = np + rv - p\mu$ ,

$$\left(\frac{rv - n}{p}\right)$$

vanishes since  $p|rv - n$  and we obtain

COROLLARY 1. Put  $\Delta = p^2\delta + rv$ . Then for all integral  $n$ ,

$$(6) \quad p_r(np^3 + \Delta) - cp_r(np + \delta) + p^{\tau-2} p_r\left(\frac{n - \mu}{p}\right) = 0.$$

This identity is equivalent to the statement that the functions  $1, F(r, p, p; \tau), F(r, p, p^3; \tau)$  are linearly dependent. Another expression for  $c$ , obtained by choosing  $n = 0$  in (6), is

$$c = \frac{p_r(\Delta)}{p_r(\delta)}.$$

We also have

COROLLARY 2. If  $n - rv$  is not divisible by  $p^2$  then

$$p_r(np^2 + rv) = \gamma_n p_r(n).$$

We go on now to some applications of Theorem 1. Suppose that  $r \geq 5$ . Then  $\gamma_n \equiv c \equiv p_r(rv) \pmod{p}$ , so that

$$(7) \quad p_r(np^2 + rv) \equiv p_r(rv) p_r(n) \pmod{p}, \quad r \geq 5.$$

We choose  $r = 11, p = 13$  in (7) as a significant example. Then from (4),  $p_r(rv) = p_{11}(77) = -16257 \equiv 6 \pmod{13}$ , so that

$$(8) \quad p_{11}(13^2n + 77) \equiv 6p_{11}(n) \pmod{13}.$$

It is known (5; 8) that

$$(9) \quad p(13n + 6) \equiv 11p_{11}(n) \pmod{13}.$$

Combining (8) and (9), we obtain the following congruence for the partition function mod 13, already given in (5):



COROLLARY 3. If  $n \equiv 6 \pmod{13}$ , then

$$p(13^2n - 7) \equiv 6p(n) \pmod{13}.$$

We can also obtain a general congruence mod  $p$  from (7), similar to those given in (5; 6).

THEOREM 2. Suppose that  $r \geq 5$ . Let  $q$  be an arbitrary integer, and set  $R = qp^2 + r$ . Then for all integral  $n$ ,

$$(10) \quad p_R(np^2 + rv) \equiv p_r(rv)p_{q+r}(n) \pmod{p}.$$

*Proof.* We have

$$\begin{aligned} \sum p_R(n)x^n &= \prod (1 - x^n)^{qp^2+r} \\ &\equiv \prod (1 - x^{np^2})^q(1 - x^n)^r \end{aligned} \pmod{p}.$$

Thus

$$p_R(n) \equiv \sum_{0 \leq k \leq n/p^2} p_q(k)p_r(n - p^2k) \pmod{p}.$$

Replace  $n$  by  $np^2 + rv$ . Since  $rv < p^2$ , we obtain

$$p_R(np^2 + rv) \equiv \sum_{k=0}^n p_q(k)p_r((n - k)p^2 + rv) \pmod{p}.$$

Formula (7) now implies that

$$p_R(np^2 + rv) \equiv p_r(rv) \sum_{k=0}^n p_q(k)p_r(n - k) \pmod{p},$$

so that  $p_R(np^2 + rv) \equiv p_r(rv)p_{q+r}(n) \pmod{p}$ , which is just (10).

As another application we prove

THEOREM 3. For all odd  $n$ ,

$$(11) \quad p_{15}(53n^2 + \frac{5}{8}(n^2 - 1)) = 0.$$

*Proof.* The proof is by induction on the total number of prime factors of  $n$ . For  $n = 1$ , (11) states that  $p_{15}(53) = 0$ , which is actually the case (4). Suppose (11) proved for all integers with not more than  $t$  prime factors. Let  $p$  be an odd prime. Then if  $n$  has precisely  $t$  prime factors, it will suffice to prove (11) for  $pn$ . Put

$$a_n = 53n^2 + \frac{5}{8}(n^2 - 1).$$

Then

$$a_{pn} = p^2a_n + \frac{5}{8}(p^2 - 1),$$

and Theorem 1 implies (with  $r = 15$ ) that  $p_{15}(a_{pn})$  is linear in  $p_{15}(a_n)$  and  $p_{15}(a_n/p)$ . Now  $p_{15}(a_n)$  vanishes by the induction hypothesis, and so does

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$p_{15}(a_{n/p})$  if  $p|n$ . If  $p \nmid n$ , however,  $a_{n/p}$  is not an integer (since 429 is square-free) and so  $p_{15}(a_{n/p})$  vanishes in this instance as well. Thus  $p_{15}(a_{pn}) = 0$  and the proof is complete.

We now prove

**THEOREM 4.** *Suppose that  $a$  is such that for the mod  $m$ ,  $p_r(a) \equiv 0 \pmod{m}$ . Suppose further that  $24a + r$  is square-free. Then*

$$(12) \quad p_r(an^2 + \frac{r}{24}(n^2 - 1)) \equiv 0 \pmod{m},$$

where  $(n, 2) = 1$  if  $3|r$  and  $(n, 6) = 1$  otherwise.

*Proof.* As in Theorem 3, the proof is by induction on the total number of prime factors of  $n$ . If  $n = 1$ , (12) states that  $p_r(a) \equiv 0 \pmod{m}$ , which is true by hypothesis. Suppose (12) proved for all integers with not more than  $t$  prime factors. Let  $p$  be a prime such that  $p > 3$  when  $(r, 3) = 1$  and  $p > 2$  otherwise. Then if  $n$  has precisely  $t$  prime factors, it will suffice to prove (12) for  $pn$ . Put

$$\lambda_n = an^2 + \frac{r}{24}(n^2 - 1).$$

Then

$$\lambda_{pn} = p^2\lambda_n + \frac{r}{24}(p^2 - 1),$$

and Theorem 1 implies that  $p_r(\lambda_{pn})$  is linear in  $p_r(\lambda_n)$  and  $p_r(\lambda_{n/p})$ . Now  $p_r(\lambda_n) \equiv 0 \pmod{m}$  by hypothesis, and the same is true for  $p_r(\lambda_{n/p})$  if  $p|n$ . If  $p \nmid n$  however,  $\lambda_{n/p}$  is not an integer since  $24a + r$  is square-free, and so  $p_r(\lambda_{n/p})$  vanishes. Thus  $p_r(\lambda_{pn}) \equiv 0 \pmod{m}$  in either case, and the proof of Theorem 4 is complete.

Theorem 4 can be strengthened slightly by discarding the condition that  $24a + r$  be square-free and restricting  $n$  to be divisible only by primes  $p$  such that  $p > 2$  when  $3|r$ ,  $p > 3$  when  $(r, 3) = 1$ , and  $p^2 \nmid 24a + r$ .

If we choose  $r = 11$ ,  $m = 13$  and  $a = 6$  we find from (4) that  $p_r(a) = p_{11}(6) = -143 \equiv 0 \pmod{13}$ , while  $24a + r = 155$  is square-free. Theorem 4 applies and we have

$$(13) \quad p_{11}(6n^2 + \frac{11}{24}(n^2 - 1)) \equiv 0 \pmod{13}, \quad (n, 6) = 1.$$

Using formula (9) once again, we obtain the following interesting congruence for the partition function mod 13:

$$(14) \quad p(84n^2 - \frac{1}{24}(n^2 - 1)) \equiv 0 \pmod{13}, \quad (n, 6) = 1.$$

Formula (14) is a Ramanujan congruence for the partition function, with the difference that the terms form a quadratic, rather than an arithmetic, progression.

More generally, we have

THEOREM 5. Suppose that  $p_{11}(a) \equiv 0 \pmod{13}$ , and that  $24a + 11$  is square-free. Then

$$(15) \quad p_{11}(an^2 + \frac{11}{24}(n^2 - 1)) \equiv 0 \pmod{13}, \quad (n, 6) = 1,$$

$$(16) \quad p((13a + 6)n^2 - \frac{1}{24}(n^2 - 1)) \equiv 0 \pmod{13}, \quad (n, 6) = 1.$$

The first few admissible  $a$ 's are 6, 10, 17, 18, 24, 27, 57, 68, 69, 74, 90, 95. (This information is extracted from (4).) It is of interest to note that two progressions

$$\left\{ a_1 n^2 + \frac{11}{24}(n^2 - 1) \right\}, \quad \left\{ a_2 n^2 + \frac{11}{24}(n^2 - 1) \right\}$$

or

$$\left\{ (13a_1 + 6)n^2 - \frac{1}{24}(n^2 - 1) \right\}, \quad \left\{ (13a_2 + 6)n^2 - \frac{1}{24}(n^2 - 1) \right\}$$

have no integers in common, since  $24a_1 + 11$  and  $24a_2 + 11$  are square-free.

III. In this section Table I gives  $p_r(rv)$  for  $r$  odd,  $5 \leq r \leq 23$  and for  $3 \leq p \leq 23$ . We exclude  $r = 1, 3$  from the table since  $p_1(n), p_3(n)$  are known explicitly. For  $p = 3$  there is no entry unless  $r$  is a multiple of 3. Using Table I we can construct Table II of values of  $c$ , and we do so for  $r$  odd,  $5 \leq r \leq 23$  and for  $p = 3, 5, 7$ . The values of  $p_r(rv)$  were extracted from (4) and some

TABLE II

$r \backslash p$	3	5	7
5		-6	16
7		66	-176
9	-12	-210	-1016
11		-2694	3544
13		11730	50008
15	1836	3990	4 23432
17		1 14810	30 34528
19		-6 45150	-39 74432
21	53028	-55 56930	444 96424
23		232 45050	13229 77768

unpublished tables in the author's possession giving the first 1000 coefficients of  $p_r(n)$  for  $r$  odd,  $5 \leq r \leq 23$ . These were computed by means of a double precision program on the IBM 704 of the National Bureau of Standards in Washington, D.C.



that  $24a + 11$  is square-

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and 13),  $(n, 6) = 1.$

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$$\left. \begin{aligned} & \\ & \end{aligned} \right\} (n^2 - 1)$$

$$\left. \begin{aligned} & \\ & \end{aligned} \right\} n^2 - \frac{1}{24}(n^2 - 1)$$

$24a_2 + 11$  are square-

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 $p_1(n), p_3(n)$  are known  
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 for  $r$  odd,  $5 \leq r \leq 23$   
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TABLE I.

$p$ $r$	3	5	7	11	13	17	19	23
5		-6	9	1	51	109	-1	191
7		41	-176	-181	-489	125	-1317	71
9	-12	-85	-673	-2423	-815	19619	46799	-80879
11		-2069	1143	29580	-16257	18413	7 64763	-13 92841
13		8605	33201	-3 70369	13 32566	65 29535	-53 77503	241 60657
15	1107	3990	3 15783	1 51789	-60 51657	847 06867	2901 77621	4580 60567
17		36685	22 10985	-839 64529	-1671 14351	-9976 89762	-40401 43705	1 17297 45647
19		-2 54525	17 90369	-7144 27549	-46099 30593	1 24427 50399	5 38898 77060	50 05958 51255
21	33345	-36 03805	444 96424	-39627 27241	-2 25204 73725	-60 37673 75677	98 99329 74081	-157 80000 80689
23		134 79425	10405 02519	-8 32370 19227	33 04666 24117	-467 07919 73011	3758 17200 49059	-21686 12339 64744

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## 1. Introd

$A = (a_{ik})$  is negative eigenvalue with a corresponding eigenvector  $x$  (unique,  $e_i > 0$ ) has been written (4, 12) where placed the p leaving invar

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The author Bonsall (1, 2) certain results (2; cf. pp. 14

2. By  $A \geq$ 

Similarly a v... as all  $x_i \geq 0$   $sp(A)$ . The fo

(I) If  $A \geq$

(II) If  $A \geq$

$n \geq 1$  is pos

(III) If  $A \geq$

$(A - \lambda I)^{-1} ($

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