Numbers n with Symmetric Count $c_n = 2$ and the Two Regions Meeting at the Center of the Dyck Path

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THEOREM:

For every number $n \in \mathbb{N}$:
$c_n = 2$ and the two regions meet at the center of the Dyck path
$\Leftrightarrow$ $n = q \times (2 \times q + 1)$, where $q \in$ A174973 and $2 \times q + 1$ is a prime .

PROOF:

"$\Leftarrow$": By the Theorem in A239929 it remains to show only that the two regions meet at the center. Observe that $r_n = 2 \times q$ and that therefore prime $2 \times q + 1$ is represented as a 1 in position $2 \times q$ in the n-th row of A237048. Therefore, $w_{n, r_n} = 0$ is the first and only zero in the n-th row of A249223, i.e., the two regions meet at the center.

"$\Rightarrow$": Let $n = q \times p$ then by the Theorem in the LINK section of A239929 we know that p is prime satisfying $2 \times q < p$. It remains to show only that $p = 2 \times q + 1$. Since the two regions meet at the center of the Dyck path we have $w_{n,k} \geq 1$, for $1 \leq k < r_n$, and $w_{n, r_n} = \sum_{j=1}^{r_n} (-1)^{j+1} \times d_{n,j} = 0$. Therefore, $d_{n, r_n} = 1$ and by Lemma 1(d) in reference in the LINK section of A241561 $r_n = 2 \times q$ since that index represents odd divisor p. With $r_n = \left\lfloor \frac{1}{2} \left(\sqrt{8 \times p \times q + 1} - 1 \right) \right\rfloor$ we get

$$2 \times q \leq \frac{1}{2} \left(\sqrt{8 \times p \times q + 1} - 1 \right) < 2 \times q + 1$$

$$\Leftrightarrow \quad (2 \times q + 1) \leq p < 2 \times q + 1 + 2 + \frac{1}{q}$$

In other words, $p = 2 \times q + 1$ or $p = 2 \times q + 3$.
Finally, since the Dyck paths for n and n-1 only meet at the diagonal the Dyck path for n has one more leg than the one for n-1. Therefore, $r_{n-1} + 1 = r_n$ and n is a triangular number, i.e., $p = 2 \times q + 1$.

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