Polygonal and Pyramidal numbers

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1 Decomposition of polygonal numbers

In this section, we will formulate and prove a proposition that will enable us to obtain the integers sequences associated with each polygonal number.

Proposition

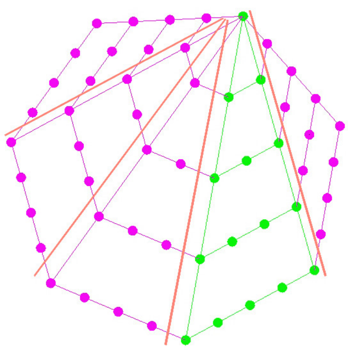
Each k-gonal number can be decomposed into the sum of a triangular number T(n) plus k-3 triangular numbers T(n-1):

$$k - gonal(n) = T(n) + (k-3)T(n-1) \tag{1}$$

Inversely, each k-gonal number is given by the sum of a triangular number T(n) plus k-3 triangular numbers T(n-1).

Proof

Consider a regular polygon of k sides, that we associate to the figurate $k - gonal(n)$ number. From a vertex of that polygon, track the joining line with other vertices, as in the heptagon example [1] shown in the next figure:



Are thus obtained $k-2$ triangular areas. If to any of these areas we associate the triangular number $T(n)$, each of the other $k-3$ remaining areas are associated with the triangular number $T(n-1)$, as can be seen by counting in the figure. Summing all triangles, we obtain the (1).

[1] This is an inductive process, so the reasoning is true for any number of sides k.

Substituting into (1) the triangular number formula: $T(n) = n(n+1)/2$, one obtains:

$$K - gonal(n) = n(n+1)/2 + (k-3)(n-1)((n-1)+1)/2$$

that factored becomes:

$$k - gonal(n) = n[(k-2)n - (k-4)]/2 \tag{2}$$

This is the same formula that appears on Wolfram MathWorld, page polygonal number, pos. (5), as it should be.

Formula (2) allows us to obtain all k-gonal numbers. We used it to get the following integer sequences, not yet present [2] in the OEIS database:

25-gonal number: $a(n) = n(23n - 21)/2$

26-gonal number: $a(n) = n(12n - 11)$

27-gonal number: $a(n) = n(25n - 23)/2$

28-gonal number: $a(n) = n(13n - 12)$

29-gonal number: $a(n) = n(27n - 25)/2$

30-gonal number: $a(n) = n(14n - 13)$

Formula (1) allows us to construct the following *table of relations* between figured k-gonal numbers:

Polygonal numbers							
Triangular (Tn)							
Square	=	Triangular + T(n-1)					
Pentagonal	=	Square + T(n-1)	=	Triangular + 2T(n-1)			
Hexagonal	=	Pentagonal + T(n-1)	=	Square + 2T(n-1)	=	Triangular + 3T(n-1)	
Heptagonal	=	Hexagonal + T(n-1)	=	Pentagonal + 2T(n-1)	=	Square + 3T(n-1)	
Octagonal	=	Heptagonal + T(n-1)	=	Hexagonal + 2T(n-1)	=	Pentagonal + 3T(n-1)	
Enneagonal	=	Octagonal + T(n-1)	=	Heptagonal + 2T(n-1)	=	Hexagonal + 3T(n-1)	
Decagonal	=	Enneagonal + T(n-1)	=	Octagonal + 2T(n-1)	=	Heptagonal + 3T(n-1)	= etc.
Hendecagonal	=	Decagonal + T(n-1)	=	Enneagonal + 2T(n-1)	=	Octagonal + 3T(n-1)	
Dodecagonal	=	Hendecagonal + T(n-1)	=	Decagonal + 2T(n-1)	=	Enneagonal + 3T(n-1)	
etc.		etc.		etc.		etc.	

The table continue indefinitely in both directions. From it you can also derive, by performing substitutions, many other relationships, such as:

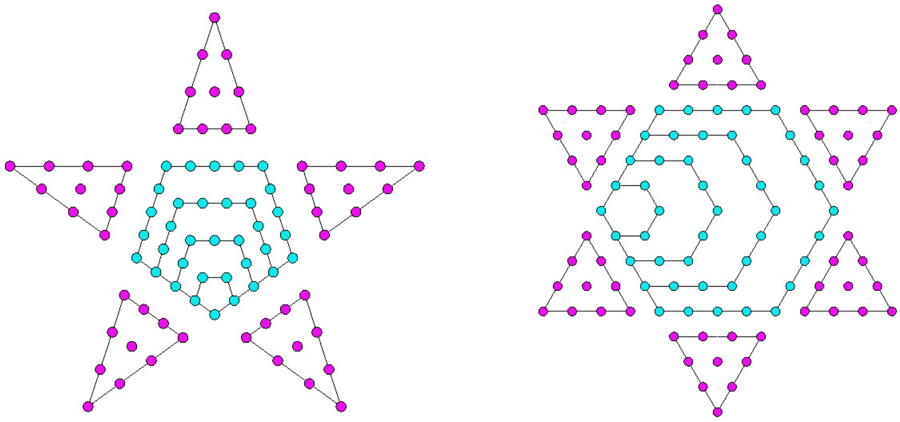
$$\text{Dod}(n) = \text{Enn}(n) - \text{Hep}(n) + \text{Hex}(n) + \text{Pen}(n) - \text{Tri}(n) + 2T(n-1)$$

[2] The 28-gonal number sequence A161935 is in the database, but with different name and meaning.

2 Star k-gonal numbers

We will show in this section how you can create, using polygonal numbers, a particular set of figurate numbers having star shape.

We will build these figurate numbers by placing on the sides of a k-gonal number, k triangular numbers, as in the cases $k = 5$ and $k = 6$ shown in the following figure:



As can be seen, the triangular numbers to be placed on the sides of polygons they must be always of the $n - 1$ order.

In our research, we just build figurate numbers using polygonal numbers from the 5-gonal to the 12-gonal, thus obtaining integer sequences, all stored in the OEIS database, but with different name and meaning.

We then added, in the comments section of the found sequences, the following annotations:

A001107 (n) = A000326(n) + 5*A000217$(n - 1)$

A051624 (n) = A000384(n) + 6*A000217$(n - 1)$

A051866 (n) = A000566(n) + 7*A000217$(n - 1)$

A051868 (n) = A000567(n) + 8*A000217$(n - 1)$

A051870 (n) = A001106(n) + 9*A000217$(n - 1)$

A051872 (n) = A001107(n) + 10*A000217$(n - 1)$

A051874 (n) = A051682(n) + 11*A000217$(n - 1)$

A051876 (n) = A051624(n) + 12*A000217$(n - 1)$

Observations

From what we saw in the above sections, it is apparent that:

1 - any $k-$gonal number is obtained from the previous $(k-1)-$gonal number by adding to it a triangular number $T(n-1)$;

2 - by adding j times the triangular number $T(n-1)$ to a $k-$gonal number, you get the $(k+j)-$gonal number;

3 - by adding k times the triangular number $T(n-1)$ to a $k-$gonal number, you get the $2k-$gonal number, which is also the star $k-$gonal number.

3 K-gonal Pyramidal Numbers

We will formulate and prove a proposition that will enable us to obtain integer sequences associated to each k-gonal pyramidal number.

Proposition

Each k-gonal pyramidal number can be decomposed into the sum of a tetrahedral number Te(n) plus k-3 tetrahedral numbers Te(n-1):

$$k - gonal - pyramidal(n) = Te(n) + (k-3)Te(n-1) \tag{3}$$

Inversely, each k-gonal pyramidal number is given by the sum of a tetrahedral number Te(n) plus k-3 tetrahedral numbers Te(n-1).

Proof

The proof is carried out in a completely analogous way to that made in Section 1 of this article. The roles of $T(n)$ and $T(n-1)$ are carried out here by $Te(n)$ and $Te(n-1)$ respectively. Vertices become edges, sides become faces, triangles become pyramids. A graphic representation is difficult to achieve, but easy to imagine: see to a succession of layers stacked to form a pyramid[3], each of which represents the number k-gonal(n) $(n = 1, 2, ..., n)$. Reasoning is the same.
Substituting in (3) the tetrahedral number formula: $Te(n) = n(n+1)(n+2)/6$, one obtains:

$$K - gonal - pyramidal(n) = n(n+1)(n+2)/6 + (k-3)n(n-1)(n+1)/6$$

that factored becomes:

$$K - gonal - pyramidal(n) = n(n+1)[(k-2)n + (5-k)]/6 \tag{4}$$

This is the same formula that appears on Wolfram MathWorld, page Pyramidal Number, pos. (1), as it should be.

[3] Note that k-gonal pyramidal numbers are given by the first partial sums of k-gonal numbers. So then: $kgp(n) = \sum kg(n) = \sum T(n) + (k-3)\sum T(n-1) = (3)$.

Formula (4) allows us to obtain all k-gonal pyramidal numbers. We used it to get the following integer sequences, not yet present in the OEIS database :

25-gonal pyramidal number: $a(n) = n(n+1)(23n-20)/6$

26-gonal pyramidal number: $a(n) = n(n+1)(8n-7)/2$

27-gonal pyramidal number: $a(n) = n(n+1)(25n-22)/6$

28-gonal pyramidal number: $a(n) = n(n+1)(26n-23)/6$

29-gonal pyramidal number: $a(n) = n(n+1)(9n-8)/2$

30-gonal pyramidal number: $a(n) = n(n+1)(28n-25)/6$

Even here, as was done at the end of Section 1, it would be possible to organize a table to derive complex relationships between k-gonal pyramidal numbers.

Observations

From what we saw above, it is apparent that:

1 - any $k-$gonal pyramidal number is obtained from the previous $(k-1)-$gonal pyramidal number by adding to it a tetrahedral number $Te(n-1)$;

2 - by adding j times the tetrahedral number $Te(n-1)$ to a $k-$gonal pyramidal number, you get the $(k+j)-$gonal pyramidal number.

4 Conclusions

Concluding this brief structural analysis, we can say that:

1 - Using unit bricks $T(n-1)$, we can construct, around the basic number $T(n)$, all the possible k-gonal and star k-gonal numbers.

2 - Likewise, in 3D space, using unit bricks $Te(n-1)$, we can construct, around the basic number $Te(n)$, all the possible k-gonal pyramidal numbers.